The Rank Effect and Market Efficiency

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February 26, 2018

A Simple Market

- Consider a market with N assets whose prices are $p_1(t), \ldots, p_N(t)$
- Suppose that all assets have the same expected growth rate:

For all $i = 1, \ldots, N, \quad \log p_i(t+1) - \log p_i(t) = \gamma + \sigma\,(B_i(t+1) - B_i(t))$

  - For all $B_1, \ldots, B_N$, the increments $B_i(t+1) - B_i(t)$ are independent standard normal r.v.'s, and $\sigma > 0$

- What happens to the prices of the N assets as time $t \to \infty$?
- Divergence: one asset grows arbitrarily more expensive than all others

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Divergence

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where B_i increments are independent standard normal r.v.'s, and $\sigma > 0$.

Let $p_{(k)}(t)$ denote the price of the k-th most expensive asset at time t, for $k = 1, \ldots, N$, so that

$$p_{(1)}(t) \geq p_{(2)}(t) \geq \cdots \geq p_{(N)}(t).$$

In the limit as $t \to \infty$, the time-averaged value of $\frac{p_{(1)}(t)}{p_{(1)}(t)+\cdots+p_{(N)}(t)} \to 1$ with probability one. But this is pretty hard to show.

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Divergence in Continuous Time

In order to prove divergence, it is necessary to work in continuous time:

$$\log p_i(t+1) - \log p_i(t) = \gamma + \sigma\left(B_i(t+1) - B_i(t)\right), \quad (1)$$

$$d\log p_i(t) = \gamma\, dt + \sigma\, dB_i(t). \quad (2)$$

In (2), B_i is a standard Brownian motion, meaning that $B_i(t+\Delta) - B_i(t)$ is normally distributed with mean zero and standard deviation Δ, for all $\Delta > 0$. In other words, (1) is a discrete-time representation of (2).

Using the continuous time setup (2), it can be shown that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{p_{(1)}(t)}{p_{(1)}(t) + \cdots + p_{(N)}(t)}\, dt = 1,$$

with probability one (Fernholz and Fernholz, JEDC 2014).

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Divergence Simulated

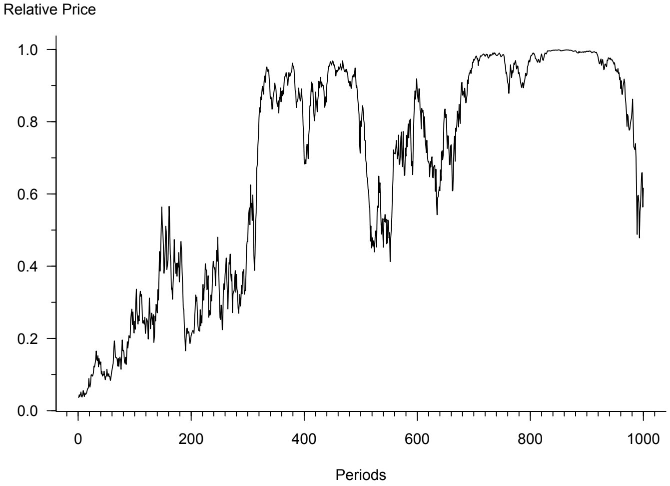

Figure: Ratio of the top-ranked price relative to all prices over time.

What are the Implications of Divergence?

- A divergent market seems unrealistic and unreasonable
- How is it, then, that markets in the real world do not diverge?
  1. Dividends
  2. Asset entry/exit
  3. Lower expected growth rates at the higher ranks
- The rank effect
  - Higher-ranked, higher-priced assets must necessarily have their prices grow more slowly than lower-ranked, lower-priced assets
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Basics

- Market consists of N assets, time $t \in [0, \infty)$ is continuous
- Price of each asset given by process p_i:

$$d \log p_i(t) = \mu_i(t)\, dt + \delta_i(t)\, dB_i(t) \tag{3}$$

  - Same setup as before, except that now expected growth rates, μ_i, and volatilities, δ_i, are potentially time-varying and can differ across assets

- Recall that (3) can be represented in discrete time as

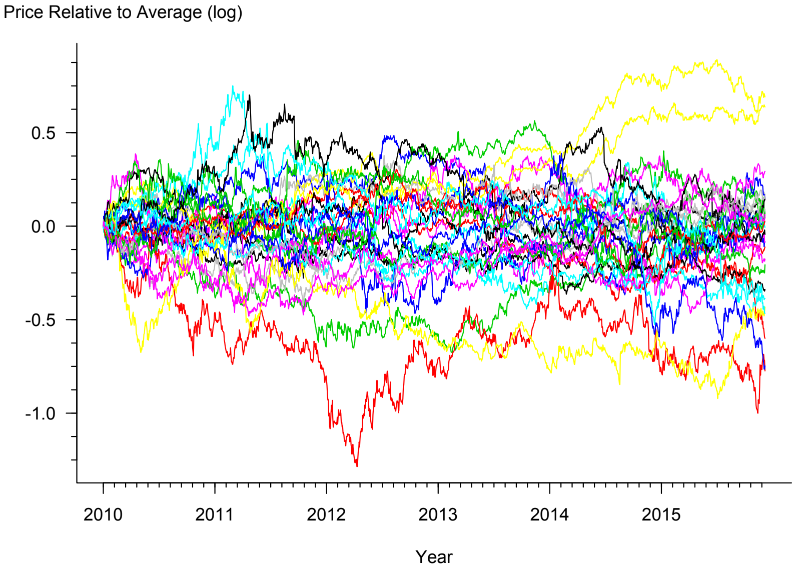
$$\log p_i(t + \Delta) - \log p_i(t) = \mu_i(t)\Delta + \delta_i(t)\left(B_i(t + \Delta) - B_i(t)\right)$$

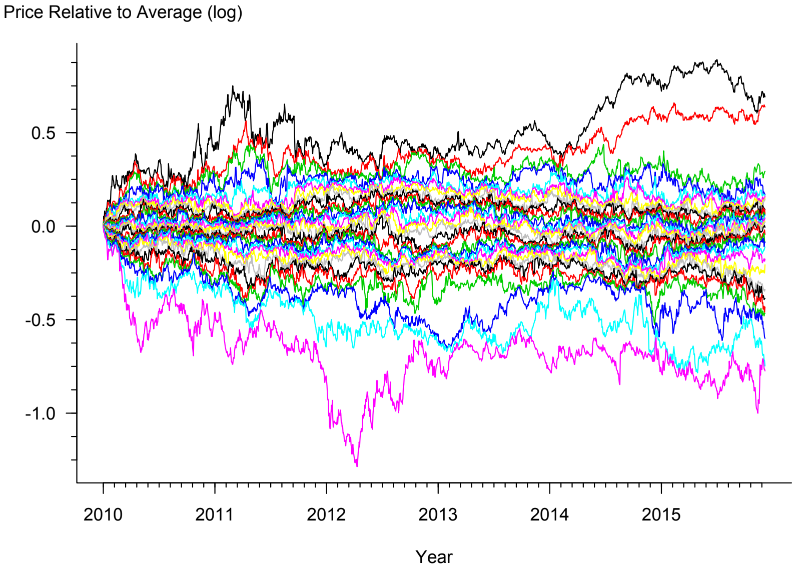
Rank-Based Price Dynamics

It is not hard to show that $p_{(k)}(t)$, the price of the k-th most expensive asset, is given by

$$d \log p_{(k)}(t) = \mu_{\omega_t(k)}(t)\, dt + \delta_{\omega_t(k)}(t)\, dB_{\omega_t(k)}(t) \ + \ \text{local time terms}$$

- $\omega_t(k) = i$ when asset i is k-th most expensive
- Local time terms measure crossovers in rank (i.e. one asset grows more expensive than another)
- For the precise definition and mathematical details, see Karatzas and Shreve, *Brownian Motion and Stochastic Calculcus* (1991)


Price Relative to Average (log)
0.5
0.0
-0.5
-1.0
2010
2011
2012
2013
2014
2015
Year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Relative Growth Rates and Volatilities

$$d\log p_{(k)}(t) = \mu_{\omega_t(k)}(t)\,dt + \delta_{\omega_t(k)}(t)\,dB_{\omega_t(k)}(t) \ + \ \text{local time terms}$$

Let α_k be the relative growth rate of the k-th ranked asset,

$$\alpha_k = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\mu_{\omega_t(k)}(t) - \mu(t)\right)\,dt,$$

where $\mu(t)$ is growth rate of all assets $p(t) = p_1(t) + \cdots + p_N(t)$.

What must be true about $\alpha_1, \ldots, \alpha_N$ in order for there to be a stable distribution of relative asset prices?

- The rank effect: $\alpha_1 + \cdots + \alpha_k < 0$, for all $k \leq N-1$

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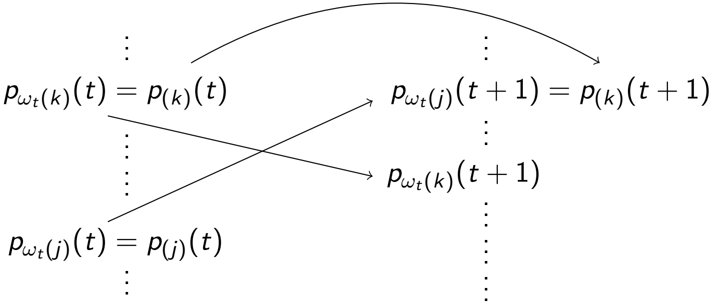
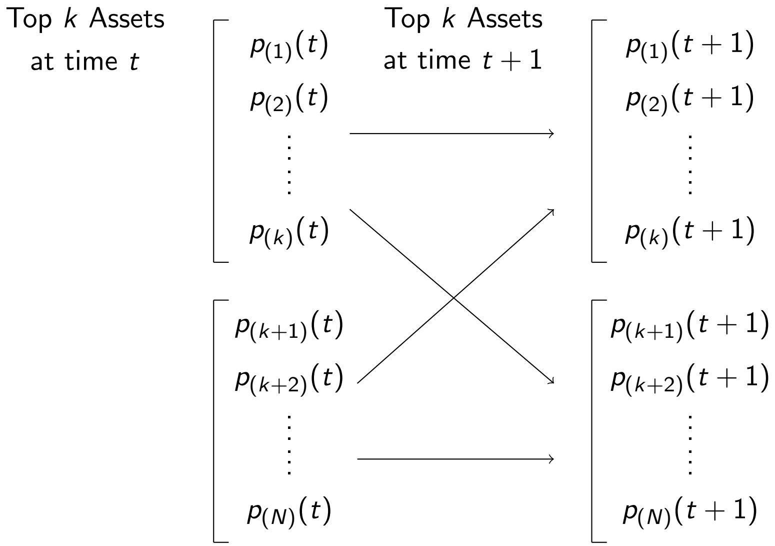
$$\begin{array}{ccc} \vdots & & \vdots \\ p_{(k)}(t) & \longrightarrow & p_{(k)}(t+1) \\ \vdots & & \vdots \\ \vdots & & \vdots \end{array}$$


$p_{\omega_t(k)}(t) = p_{(k)}(t)$
$p_{\omega_t(j)}(t) = p_{(j)}(t)$
$p_{\omega_t(j)}(t+1) = p_{(k)}(t+1)$
$p_{\omega_t(k)}(t+1)$


Top k Assets at time t
$p_{(1)}(t)$
$p_{(2)}(t)$
$p_{(k)}(t)$
$p_{(k+1)}(t)$
$p_{(k+2)}(t)$
$p_{(N)}(t)$
Top k Assets at time $t+1$
$p_{(1)}(t+1)$
$p_{(2)}(t+1)$
$p_{(k)}(t+1)$
$p_{(k+1)}(t+1)$
$p_{(k+2)}(t+1)$
$p_{(N)}(t+1)$

What Assets to Use?

- How is it that markets in the real world do not diverge?
  1. Dividends
  2. Asset entry/exit
  3. Lower expected growth rates at the higher ranks
- In what real-world asset markets are dividends and entry/exit unlikely to be major factors?
- Future and spot commodity prices
  - Daily future price of 30 commodities from 2010-2015
  - Monthly spot commodity prices of 22 commodities from 1980-2015, and 15 commodities from 1882-1913

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Relative Commodity Futures Prices by Commodity

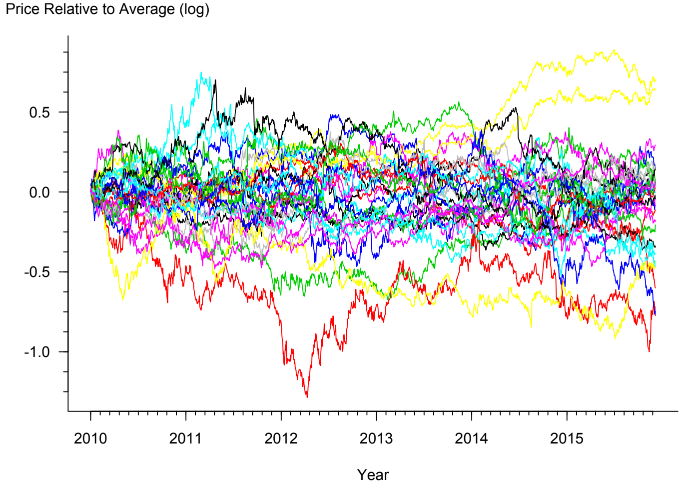

Figure: Log commodity futures prices relative to the average, 2010-2015.

Relative Commodity Futures Prices by Rank

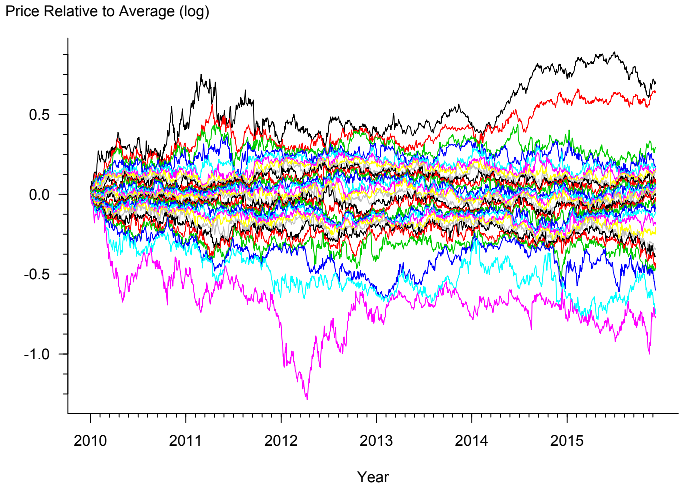

Figure: Log ranked commodity futures prices relative to the average, 2010-2015.

Returns: Low-Rank vs. High-Rank Commodities

- Consider various rank cutoffs for low- and high-rank portfolios
  - Particular emphasis on quintile, decile, and median sorts
  - Portfolios place equal weight on each commodity in the portfolio
- Lower-ranked, lower-priced commodity futures portfolios consistently outperform higher-ranked, higher-priced commodity futures portfolios
  - Quintile sort: 23.2% low-minus-high average excess yearly return
  - Quintile sort: low correlation with Russell 3000 returns
  - Median sort generates less volatile and lower excess returns

Equities, Bonds, and the Rank Effect

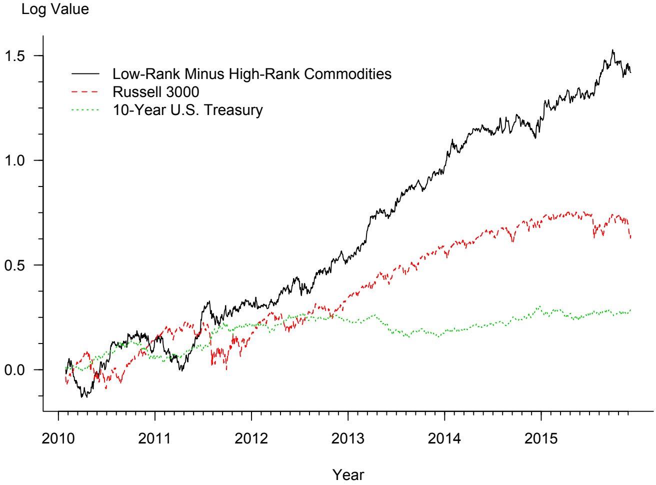

Figure: Log returns for rank effect (quintile sort), stocks, and bonds, 2010-2015.

The Rank Effect with Different Cutoffs

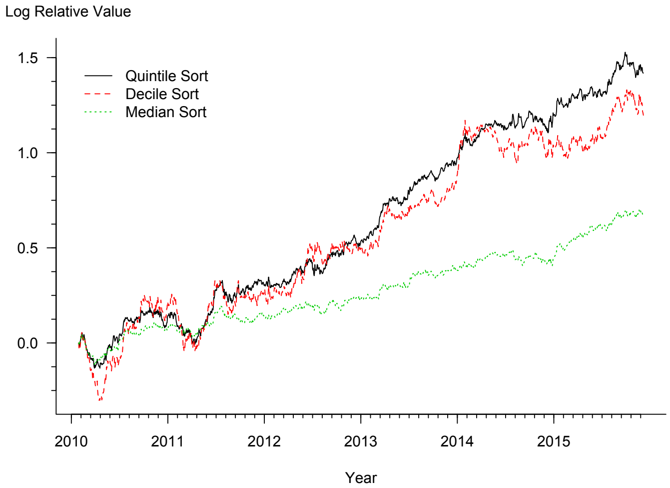

Figure: Log relative returns for rank effect portfolios, 2010-2015.

The Rank Effect with Different Cutoffs

Rank Cutoff	Intercept	Russell 3000 Daily Return (β)
15	4.24 (1.73)	0.075 (0.016)
14	4.78 (1.79)	0.078 (0.017)
13	5.75 (1.87)	0.087 (0.018)
12	5.80 (1.96)	0.092 (0.019)
11	6.59 (2.05)	0.093 (0.019)
10	7.11 (2.18)	0.102 (0.021)
9	7.13 (2.29)	0.109 (0.022)
8	7.50 (2.44)	0.109 (0.023)
7	9.35 (2.67)	0.107 (0.025)
6	9.24 (2.88)	0.104 (0.027)
5	9.32 (3.19)	0.092 (0.030)
4	8.06 (3.60)	0.076 (0.034)
3	8.08 (4.11)	0.040 (0.039)
2	7.74 (5.25)	0.018 (0.050)
1	10.90 (7.82)	-0.066 (0.074)

Table 1: Regression results using daily excess returns (basis points) of lower-ranked commodity futures portfolios relative to higher-ranked commodity futures portfolios for all 15 rank cutoffs, 2010 – 2015. Standard errors are reported in parentheses.

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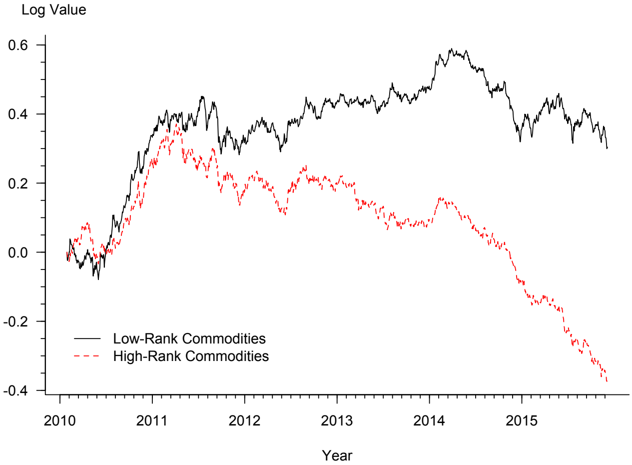
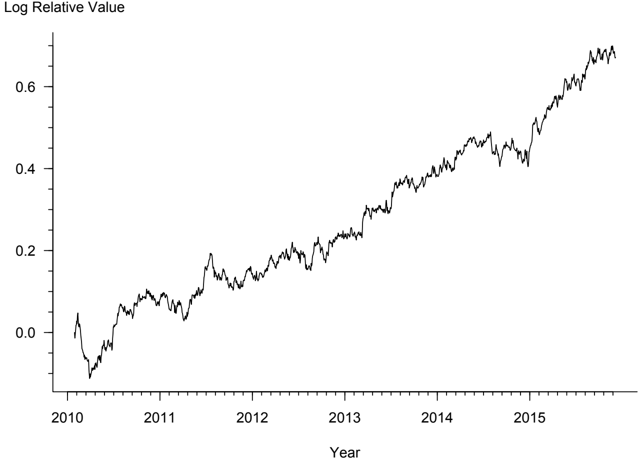


Figure: Log returns for low-rank (bottom half) and high-rank (top half) commodities portfolios, 2010-2015.

Relative Returns: Low-Rank vs. High-Rank Commodities



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Relative Spot Prices by Commodity

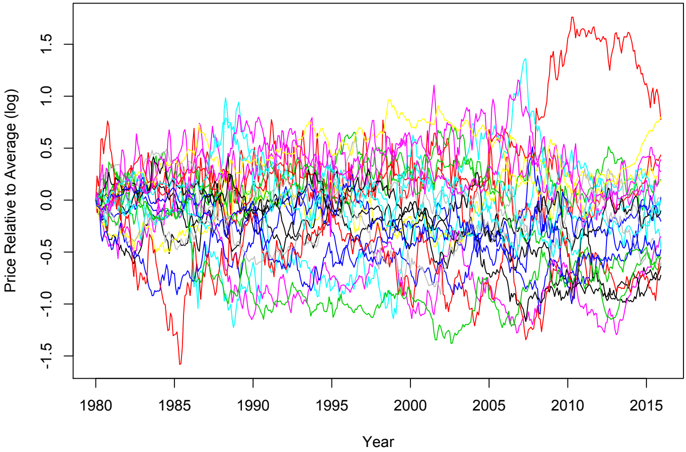


Figure: Log spot commodity prices relative to the average, 1980-2015.

Relative Spot Prices by Rank

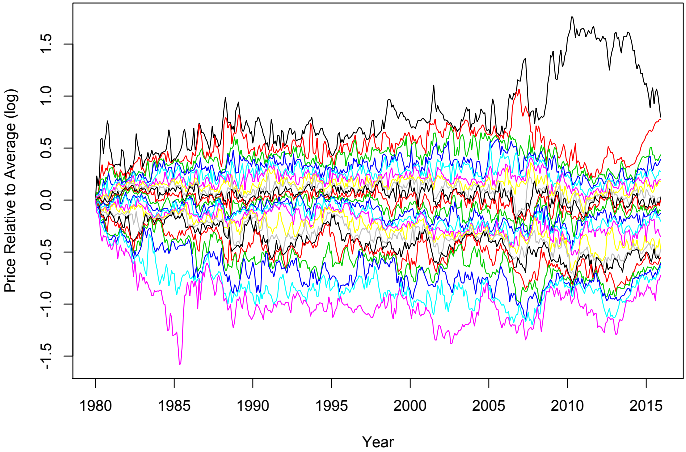
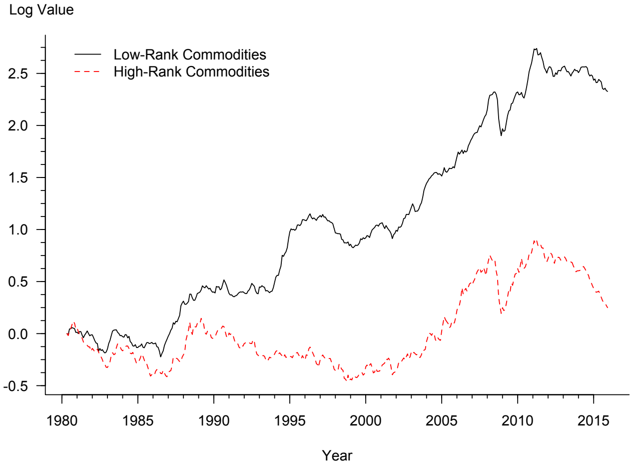


Figure: Log ranked spot commodity prices relative to the average, 1980-2015.

Returns: Low-Rank vs. High-Rank Commodities



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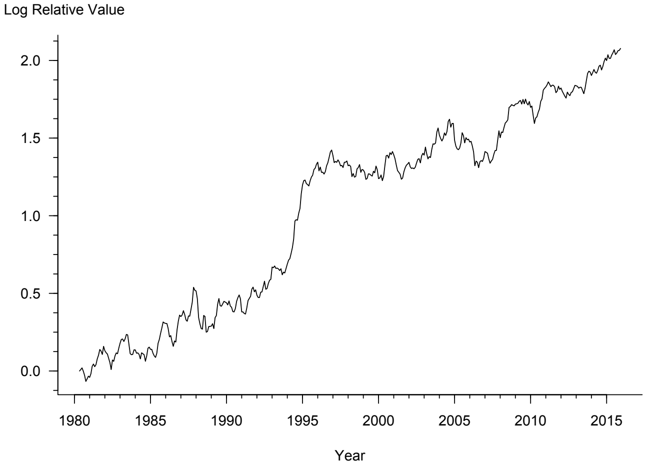

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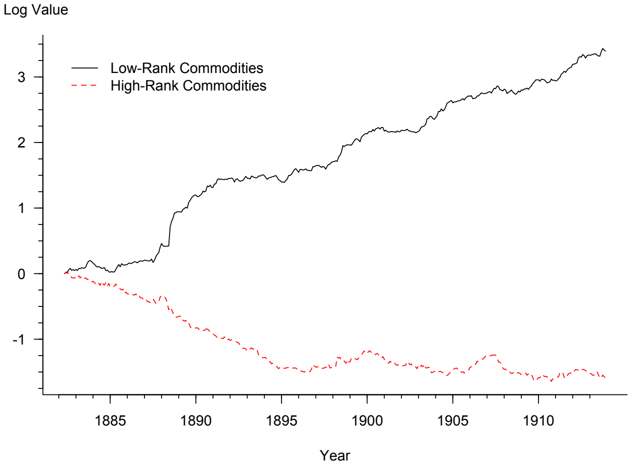

Figure: Log returns for low-rank (bottom half) and high-rank (top half) commodities portfolios, 1882-1913.

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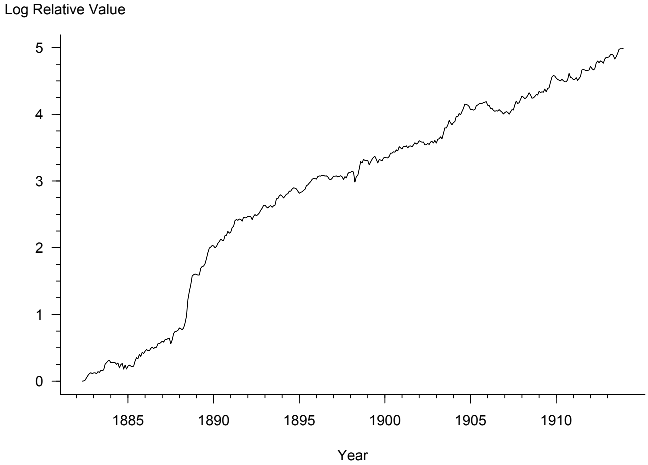

Figure: Log returns of low-rank (bottom half) portfolio relative to high-rank (top half) portfolio, 1882-1913.

Returns: Low-Rank vs. High-Rank Commodities

Once again, lower-ranked, lower-priced commodity portfolios outperform higher-ranked, higher-priced commodity portfolios

- 1980-2015: 5.7% (16.4%) low-minus-high average excess yearly return for median (quintile) sort, negative correlation with U.S. stock market returns
- 1882-1913: 15.1% (28.8%) low-minus-high average excess yearly return for median (quintile) sort

How Important is the Start Date?

- How robust are these results across different start dates?
- This is easily investigated empirically
  - Normalize prices, wait for prices to approach stationary distribution, and then form portfolios based on rank for different start dates
  - Do this with spot commodity prices from 1980-2015, since much longer time frame

Relative Returns: Varying the Start Date

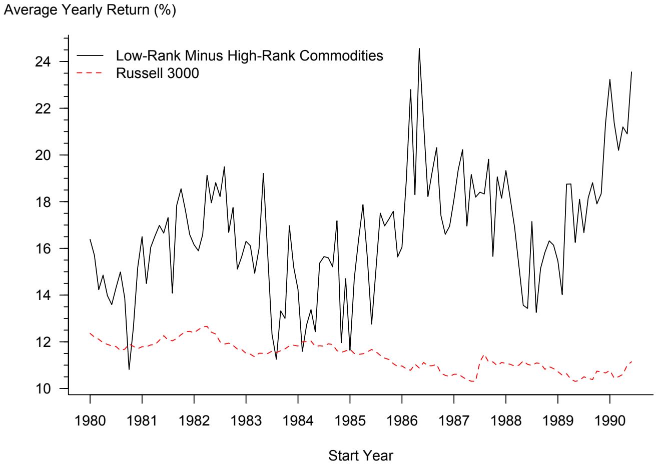

Figure: Low-minus-high rank effect returns (quintile sort) and Russell 3000 returns for different start dates, 1980-2015.

Market Correlation: Varying the Start Date

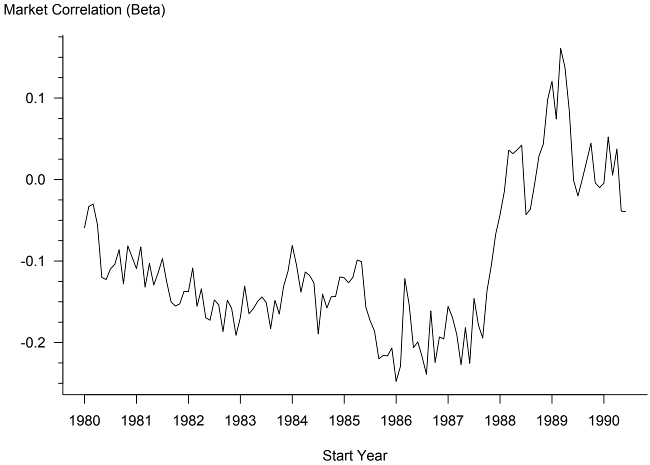

Figure: Correlation between low-minus-high rank effect returns (quintile sort) and Russell 3000 returns for different start dates, 1980-2015.

The Rank Effect in Practice

- Equal-weighted strategies of low- and high-ranked commodity futures are simple and do not require any special information
  - Much more sophisticated strategies using similar ideas exist
- In fact, rank effect for commodities was already known
  - Commodity "value": Asness, Moskowitz, and Pedersen (2013)
  - Value is considered an asset pricing factor across multiple markets
  - But what makes something a "factor"?

The Rank Effect and Commodity "Value"

- Asness, Moskowitz, and Pedersen (2013)
  - Define value as average price 4.5-5.5 years ago relative to current price
  - High-value commodity futures outperform low-value commodity futures with zero beta
- For 1980-2015, correlation between "value" returns as in Asness et al. (2013) and rank effect returns is 0.5
  - Rank effect and value generate similar, but far from identical, portfolios
- The rank effect explains why high-value commodity futures outperform low-value commodity futures

The Rank Effect and the Size Effect

- The rank effect also implies that larger stocks must generate lower capital gains than smaller stocks
  - This is the classic size effect (Banz, 1981; Fama and French, 1993)
  - Many potential explanations for the size effect (Van Dijk, 2011)
  - Of course, rank effect requires a stable distribution of total market capitalizations, which is strongly supported by the data
- The rank effect offers an alternate, structural explanation of the classic size effect
  - Complications exist, however, since stocks pay dividends and also enter (IPOs) and exit (bankruptcy)

Market Efficiency and Economic Equilibrium

- How can the rank effect be reconciled with standard notions of market efficiency and economic equilibrium?

1. Rank effect generates high returns, so everyone should exploit it
   - This, in turn, will make the rank effect go away
   - The rank effect is "arbitraged away"
2. Rank effect generates high returns, but this is compensation for risk
   - Even though the rank effect looks like a great investment strategy, it is in fact "risky"
   - Fama (1970, 1991): Any test of market efficiency is a test of efficiency together with an asset pricing model

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How Can the Rank Effect be Arbitraged Away?

- The rank effect seemingly cannot be arbitraged away
  - Relies on a stable relative price distribution
  - Relies on price heterogeneity and ranking
  - Difficult to see how actions of investors can alter prices in a way that violates these weak conditions
- Intuitive notion of good deals being "arbitraged away" by profit-seeking investors may not be applicable in some real-world cases

Rank, Risk, and Market Efficiency

- If the rank effect cannot be arbitraged away, then how can it be reconciled with market efficiency?
  - There must be a systematic relationship between rank and risk
- In economics, risk factors are linked to the utility of investors
  - Risky investments do poorly in bad times, when utility is low
- Why should lower-ranked assets be systematically riskier than higher-ranked assets?
  - How does asset rank relate to the utility of investors?

The End

Thank You