A Model of Economic Mobility and the Distribution of Wealth

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Income and Wealth Distributions

- Significant concentration of income and wealth observed worldwide
- Richest 1% holds 40% of wealth, earns 25% of income in U.S.
  - Atkinson, Piketty, and Saez (2011), Saez and Zucman (2014)
- Gini coefficients of 0.55 for income and 0.80 for wealth in U.S.
  - Díaz-Giménez, Quadrini, Ríos-Rull, and Rodríguez (2002)
  - Davies, Sandström, Shorrocks, and Wolff (2011)
- Many different explanations have been proposed
  - Krussel and Smith (1998), Quadrini (2000), De Nardi (2004)

Idiosyncratic Investment Risk

- Uninsurable idiosyncratic investment risk another possible explanation
  - Angeletos and Calvet (2006), Benhabib, Bisin, and Zhu (2011)
  - Generates realistic stationary Pareto distribution of wealth
- Strong empirical motivation for uninsurable investment risk
  - Housing: Case and Shiller (1989), Flavin and Yamashita (2002)
  - Private equity: Moskowitz and Vissing-Jorgensen (2002)
  - Together, these make up more than 50% of total U.S. household wealth: Bertaut and Starr-McCluer (2002) and Wolff (2012)

Contributions

1. Application of new rank-based solution technique (Fernholz, 2015)
   - Nonparametric approach that can be used to solve many models
   - Household-by-household solution for a simple model with idiosyncratic investment risk and intergenerational transfers

2. Detailed description and analysis of economic mobility
   - Analytic and numerical results for several measures of mobility

3. Examination of the implications of risk-sharing subgroups of households
   - Results about welfare and distributional implications

Economic Mobility and the Distribution of Wealth

$$\text{inequality} = \frac{\text{idiosyncratic investment risk}}{\text{cross-sectional mean reversion}}$$

$$\text{mobility} = \frac{\text{cross-sectional mean reversion}}{\text{inequality}}$$

- Mobility result is consistent with recent empirical work
  - "Great Gatsby curve" (Krueger, 2012; Corak, 2013)
  - Geographical variation within U.S. (Chetty et al., 2014)
- Risk-sharing subset of HHs increases welfare and wealth accumulation
  - Subgroup rises or falls in the distrib. depending on level of inequality

A Nonparametric Approach to Wealth Distribution

- Economy is populated by N households, time $t \in [0, \infty)$ is continuous
- Total wealth of each household given by process w_i:

$$d \log w_i(t) = \mu_i(t)\, dt + \sum_{z=1}^{M} \delta_{iz}(t)\, dB_z(t)$$

  - $B_1, \ldots, B_M$ are independent Brownian motions ($M \geq N$)
  - Little structure imposed on μ_i and δ_{iz}
  - Only for i.i.d.-like processes is model clearly inappropriate

Rank-Based Wealth Dynamics and Local Times

Let $w_{(k)}(t)$ be the total wealth of the k-th wealthiest household:

$$d\log w_{(k)}(t) = \mu_{p_t(k)}(t)\,dt + \sum_{z=1}^{M} \delta_{p_t(k)z}(t)\,dB_z(t) + \frac{1}{2}d\Lambda_{\log w_{(k)} - \log w_{(k+1)}}(t) - \frac{1}{2}d\Lambda_{\log w_{(k-1)} - \log w_{(k)}}(t).$$

- $p_t(k) = i$ when household i is the k-th wealthiest household
- Λ_x is the *local time* at 0 for the process x
  - Measures amount of time x spends near 0 (Karatzas and Shreve, 1991)

Let $\theta_{(k)}(t)$ be share of total wealth held by k-th wealthiest household:

$$\theta_{(k)}(t) = \frac{w_{(k)}(t)}{w(t)} = \frac{w_{(k)}(t)}{w_1(t) + \cdots + w_N(t)}.$$

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Relative Growth Rates and Volatilities

$$d\log w_{(k)}(t) = \mu_{p_t(k)}(t)\,dt + \sum_{z=1}^{M} \delta_{p_t(k)z}(t)\,dB_z(t) + \text{ local time terms}$$

Let α_k be the relative growth rate of the k-th wealthiest household,

$$\alpha_k = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\mu_{p_t(k)}(t) - \mu(t)\right)\, dt,$$

where $\mu(t)$ is growth rate of total wealth $w(t) = w_1(t) + \cdots + w_N(t)$.

Let σ_k be the volatility of relative wealth holdings,

$$\sigma_k^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{z=1}^{M} \left(\delta_{p_t(k)z}(t) - \delta_{p_t(k+1)z}(t)\right)^2\, dt.$$

Theorem (Distribution of Wealth)

There is a stable distribution of wealth in this economy if and only if $\alpha_1 + \cdots + \alpha_k < 0$*, for* $k = 1, \ldots, N-1$*. Furthermore, if there is a stable distribution of wealth, then for* $k = 1, \ldots, N-1$*, this distribution satisfies*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\log\hat{\theta}_{(k)}(t) - \log\hat{\theta}_{(k+1)}(t)\right)\, dt = \frac{\sigma_k^2}{-4(\alpha_1 + \cdots + \alpha_k)}.$$

- Stable distribution entirely determined by two factors
  1. Volatility of relative wealth holdings: σ_k^2
  2. Cross-sectional mean reversion: $-\alpha_k$
- Theorem describes behavior of stable versions of wealth shares, $\hat{\theta}_{(k)}$

Heterogeneous Households and Idiosyncratic Risk

- Economy is populated by N dynastic households that live for S years
  - Choose savings-consumption for $t \leq S$, and choose bequest for $t = S$
- Households have two investment options:
  1. Risk-free asset that pays a return of r
  2. Individual-specific asset subject to idiosyncratic, uninsurable risk

  For all $i = 1, \ldots, N$, price of individual-specific risky asset given by

$$dP_i(t) = \lambda P_i(t)\, dt + \kappa P_i(t)\, dB_i(t)$$

  - Angeletos & Panousi (2011): producivity shocks for private business

Household Maximization Problems

Each household i solves the problem:

$$J(w,t) = \max_{c_i,\phi_i} E_t \left[\int_t^S e^{-\rho(s-t)} \frac{c_i^{1-\gamma}(s)}{1-\gamma}\, ds + e^{-\rho(S-t)} \chi \frac{((1-\tau)w_i(S))^{1-\gamma}}{1-\gamma} \right]$$

$$\text{s.t.} \quad dw_i(s) = \left[rw_i(s) + (\lambda - r)\phi_i(s)w_i(s) - c_i(s) \right] ds + \kappa\phi_i(s)w_i(s)\, dB_i(s)$$

$c_i(t)$: Consumption

$\phi_i(t)$: Fraction of wealth $w_i(t)$ invested in risky asset

$\tau < 1$: Estate tax rate

$\chi > 0$: Intensity of bequest motive

$\gamma \geq 1$: Coefficient of relative risk aversion

Proposition (Consumption, Investment, and Bequests)

For each household $i = 1, \ldots, N$ *and at each point in time* $0 \le t \le S$*, the policy functions* $\phi_i(t)$ *and* $c_i(t)$ *are given by*

$$\phi_i(t) = \frac{\lambda - r}{\gamma\kappa^2},$$

$$c_i(t) = \left(\frac{e^{\eta(S-t)} - 1}{\eta} + \left(\chi(1-\tau)^{1-\gamma}\right)^{\frac{1}{\gamma}} e^{\eta(S-t)}\right)^{-1} w_i(t),$$

where

$$\eta = \frac{(1-\gamma)r - \rho}{\gamma} + \frac{(1-\gamma)(\lambda - r)^2}{2\gamma^2\kappa^2}.$$

Standard intuition behind optimal household behavior (Merton, 1969)

Household Wealth Dynamics

In equilibrium, each household $i = 1, \ldots, N$ has wealth dynamics that follow

$$d \log w_i(t) = \psi(t)\, dt + \left(\frac{\lambda - r}{\gamma\kappa}\right) dB_i(t),$$

where $0 \leq t \leq S$, and

$$\psi(t) = r + \frac{(2\gamma - 1)(\lambda - r)^2}{2\gamma^2\kappa^2} - \left(\frac{e^{\eta(S-t)} - 1}{\eta} + (\chi(1-\tau)^{1-\gamma})^{\frac{1}{\gamma}} e^{\eta(S-t)}\right)^{-1}.$$

At the end of its life (at time $t = S$), each household i leaves an after-tax bequest of $(1 - \tau)w_i(S)$ to its newborn offspring.

Fiscal Transfers to Newborn Offspring

- Model as it is setup so far has no stable distribution of wealth
  - Gabaix (1999, 2009), Fernholz and Fernholz (2014)
- Stable distribution requires some mechanism of mean reversion
  - Fiscal transfer to poorest households (Benhabib et al., 2014)
  - Finite lifespans and labor income (Benhabib et al., 2011)
- Introduce fiscal policy in which government provides newborn households with a single lump-sum transfer
  - Each newborn household i receives $\nu_i(t)$, where $0 \leq \nu_i(t) \leq \frac{\tau w(t)}{N}$

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Discontinuous Wealth Processes

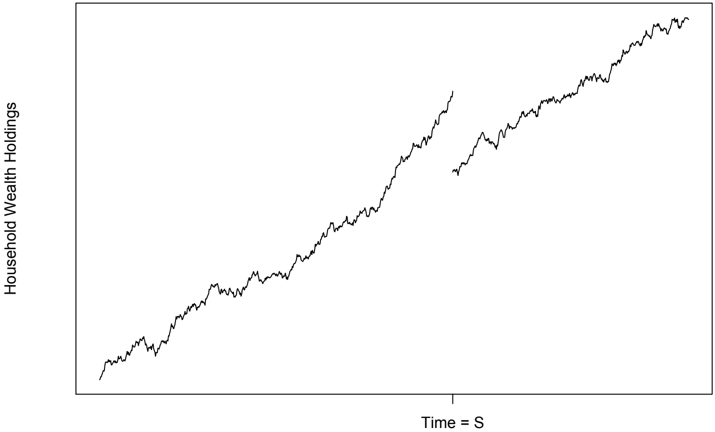

Figure: Because of estate tax and lump-sum transfers, wealth processes are discontinuous at times of intergenerational transfers.

Continuous Wealth Processes

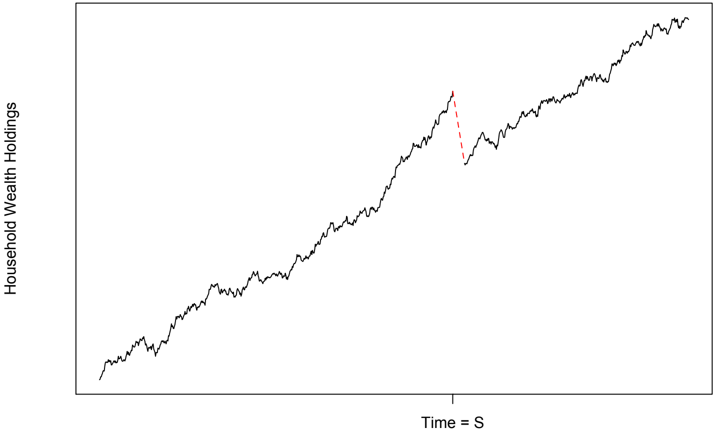

Figure: Solution: Change discontinuous wealth processes w_i into continuous wealth processes $\tilde{w}_i$.

Application: General Solution Techniques

Recall the general characterization of the stable wealth distribution,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\log \hat{\theta}_{(k)}(t) - \log \hat{\theta}_{(k+1)}(t) \right) dt = \frac{\sigma_k^2}{-4(\alpha_1 + \cdots + \alpha_k)},$$

where

$$\alpha_k = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\mu_{p_t(k)}(t) - \mu(t) \right) dt,$$

$$\sigma_k^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{z=1}^{M} \left(\delta_{p_t(k)z}(t) - \delta_{p_t(k+1)z}(t) \right)^2 dt.$$

To solve the model, then, it is necessary to determine α_k and σ_k for the continuous wealth processes $\tilde{w}_i$.

Rank-Based Relative Growth Rates and Volatilities

$$\tilde{\alpha}_k = \lim_{j \to \infty} \frac{1}{S} \log \left(1 - \tau + \frac{\nu_{(k)}(jS)}{\tilde{w}_{(k)}(jS)} \right) - \frac{1}{SN} \sum_{\ell=1}^{N} \log \left(1 - \tau + \frac{\nu_{(\ell)}(jS)}{\tilde{w}_{(\ell)}(jS)} \right)$$

$$\tilde{\sigma}_k = \tilde{\sigma} = \sqrt{2} \left(\frac{\lambda - r}{\gamma \kappa} \right)$$

- Cross-sectional mean reversion, as measured by $-\tilde{\alpha}_k$, depends on how redistributive lump-sum transfers to newborn offspring are
- Household exposure to idiosyncratic investment risk, $\tilde{\sigma}$, depends on risk-adjusted excess return of individual-specific assets

Theorem (Equilibrium Distribution of Wealth)

There exists a steady-state distribution of wealth in this economy if and only if $\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k < 0$*, for* $k = 1, \ldots, N-1$*. Furthermore, if there is a steady-state distribution of wealth, then for* $k = 1, \ldots, N-1$*, this distribution satisfies*

$$E\left[\log \theta^*_{(k)}(t) - \log \theta^*_{(k+1)}(t)\right] = \frac{\tilde{\sigma}^2}{-4(\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k)}, \quad \text{a.s.}$$

Equilibrium distribution of wealth depends on two factors

1. Cross-sectional mean reversion: $-\tilde{\alpha}_k$
2. Household exposure to idiosyncratic investment risk: $\tilde{\sigma}$

Theorem (Economic Mobility)

If there exists a steady-state equilibrium distribution of wealth, then for all $k = 1, \dots, N-1$,

$$E\left[S_k(t) \mid \theta^*_{(k)}(t), \theta^*_{(k+1)}(t)\right] = \frac{\log \theta^*_{(k)}(t) - \log \theta^*_{(k+1)}(t)}{-2(\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k)},$$

$$E\left[S_k(t)\right] = \frac{\tilde{\sigma}^2}{8(\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_k)^2}.$$

- $S_k(t)$ marks the first time after t that the $k+1$-th wealthiest household overtakes the k-th wealthiest household
- Mobility is decreasing in inequality, and increasing in mean reversion
  - Other, more common measures of mobility show same behavior

Implications of Increased Market Completeness

- Suppose that some subgroup of households can pool idiosyncratic risk
  - By symmetry, all households fully diversify across available risky assets
- This risk-sharing has two separate effects in equilibrium:
  1. Faster wealth accumulation, a mechanical conseq. of diversific.
  2. More risky investment, as risk-sharing subgroup of households changes behavior in response to better investment options
- Both effects increase welfare for all households in the economy
- If inequality is high, risk-sharing subgroup rises to the top of the wealth distribution; if inequality is low, it drops near the bottom

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Understanding the Results

$$\text{inequality} = \frac{\text{idiosyncratic investment risk}}{\text{cross-sectional mean reversion}}$$

$$\text{mobility} = \frac{\text{cross-sectional mean reversion}}{\text{inequality}}$$

- Broadly consistent with some recent empirical results
  - "Great Gatsby curve" (Krueger, 2012; Corak, 2013)
  - Geographical variation within U.S. (Chetty et al., 2014)
- Presence of a risk-sharing subset of households raises welfare for all
  - Subgroup rises or falls in the distrib. depending on level of inequality

Baseline Parameterization

$$E\left[\log\theta^*_{(k)}(t)-\log\theta^*_{(k+1)}(t)\right]=\frac{\tilde{\sigma}^2}{-2(\tilde{\alpha}_1+\cdots+\tilde{\alpha}_k)}$$

- Number of households $N = 1,000,000$, length of life $S = 50$
- Discount rate $\rho = 0.03$, bequest motive $\chi = 1$, estate tax rate $\tau = 0.2$
- Idiosyncratic volatilities: $\tilde{\sigma} = \sqrt{2}\left(\frac{\lambda - r}{\gamma\kappa}\right)$
  - $\lambda = 0.07, \kappa = 0.2, \gamma = 1.5, r = 0.03 \implies \tilde{\sigma} = 0.189$
  - Flavin & Yamashita (2002), Moskowitz & Vissing-Jorgensen (2002), Angeletos (2007), Benhabib, et al. (2011, 2014)

Baseline Parameterization

- All households receive same lump sum transfer $\nu = \nu_1 = \cdots = \nu_N$
- If we write ν as a fraction of total wealth of all households, so that $\nu = \bar{\nu}\tilde{w}$, then the parameters $\tilde{\alpha}_k$ satisfy

$$\begin{aligned}\tilde{\alpha}_k &= \lim_{j\to\infty} \frac{1}{S} \log\left(1 - \tau + \frac{\nu}{\tilde{w}_{(k)}(jS)}\right) - \frac{1}{SN}\sum_{\ell=1}^{N} \log\left(1 - \tau + \frac{\nu}{\tilde{w}_{(\ell)}(jS)}\right)\\ &= \lim_{j\to\infty} \frac{1}{S} \log\left(1 - \tau + \frac{\bar{\nu}}{\tilde{\theta}_{(k)}(jS)}\right) - \frac{1}{SN}\sum_{\ell=1}^{N} \log\left(1 - \tau + \frac{\bar{\nu}}{\tilde{\theta}_{(\ell)}(jS)}\right)\end{aligned}$$

- Choose value of $\bar{\nu}$ that most closely matches U.S. wealth distribution data from Saez and Zucman (2014)

Baseline Parameterization

Household Wealth Percent Rank	Model Wealth Shares	U.S. Wealth Shares
0-0.01	11.6%	11.2%
0.01-0.1	10.7%	10.8%
0.1-0.5	11.7%	12.5%
0.5-1	6.5%	7.3%
1-10	29.5%	35.4%
10-100	29.8%	22.8%

Table: Household wealth shares for the baseline parameterization of the model and for the 2012 U.S. wealth distribution.

Baseline Parameterization

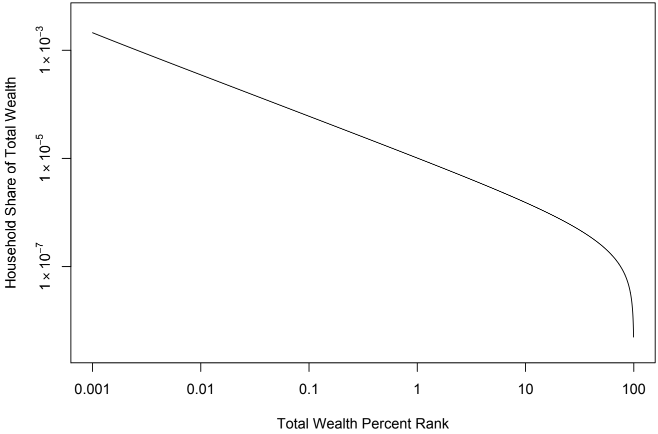

Figure: Individual households' wealth shares for the baseline parameterization of the model.

Changing Inequality

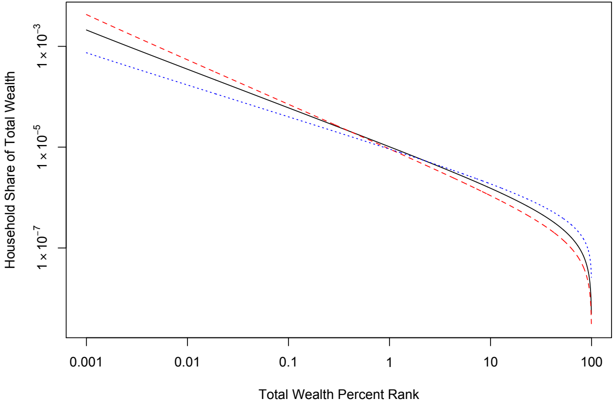

Figure: Individual households' wealth shares for three parameterizations of the model: baseline (solid black line), lower lump-sum transfer ratio (dashed red line), and lower exposure to idiosyncratic investment risk (dotted blue line).

Baseline Parameterization, Intergenerational Mobility

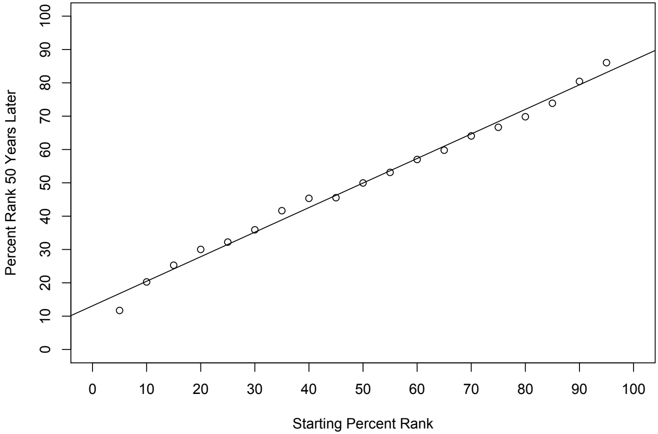

Figure: Intergenerational mobility (50 years) for the baseline parameterization of the model. The rank-rank slope and intercept are 0.737 and 12.69, respectively.

Lower Lump-Sum Transfer Ratio

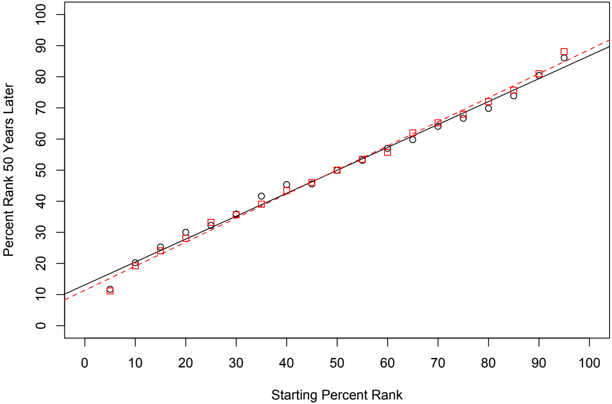

Figure: Intergenerational mobility (50 years) for two parameterizations of the model: baseline (solid black line, rank-rank slope of 0.737) and lower lump-sum transfer ratio (dashed red line, rank-rank slope of 0.765).

Lower Investment Risk

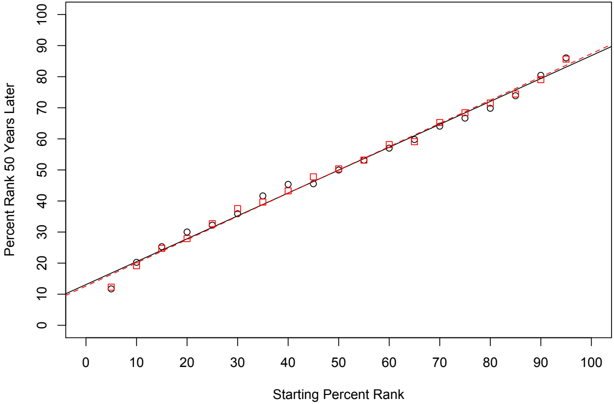

Figure: Intergenerational mobility (50 years) for two parameterizations of the model: baseline (solid black line, rank-rank slope of 0.737) and lower exposure to idiosyncratic investment risk (dashed red line, rank-rank slope of 0.760).

Lower Lump-Sum Transfer Ratio and Investment Risk

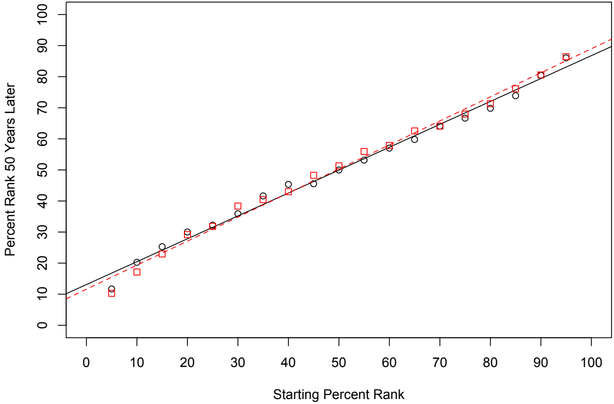

Figure: Intergenerational mobility (50 years) for two parameterizations of the model: baseline and both lower lump-sum transfer ratio and lower exposure to idiosyncratic investment risk (dashed red line, rank-rank slope of 0.787).

Baseline Parameterization, Risk-Sharing Subgroup

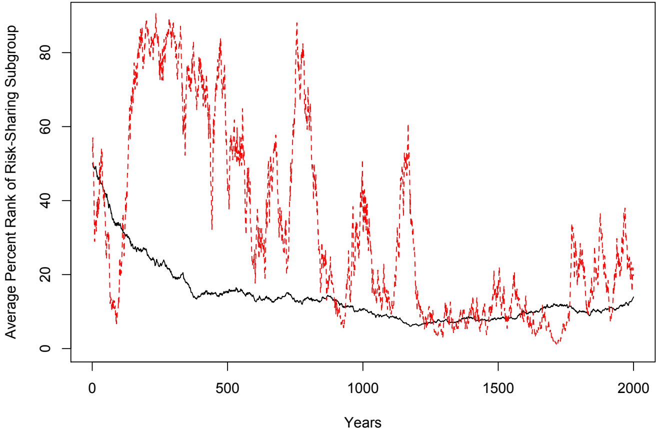

Figure: The average percent rank of the households in the risk-sharing subgroup over time for the baseline parameterization of the model.

Changing Inequality, Risk-Sharing Subgroup

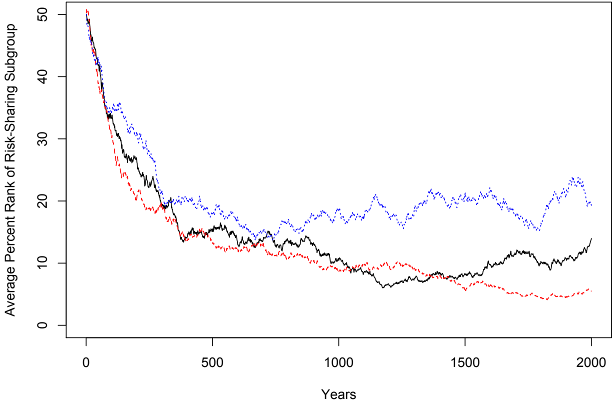

Figure: Average percent rank of risk-sharing subgroup for 3 parameterizations: baseline (solid black line), lower lump-sum transfer ratio (dashed red line), and lower exposure to idiosyncratic investment risk (dotted blue line).

Extensions and Applications

$$\text{inequality} = \frac{\text{idiosyncratic investment risk}}{\text{cross-sectional mean reversion}}$$

$$\text{mobility} = \frac{\text{cross-sectional mean reversion}}{\text{inequality}}$$

- What if households are different from each other?
  - Heterogeneous income profiles, or "skill", reduces mobility
- What about purely empirical applications of the solution techniques?
  - U.S. wealth distribution (Fernholz, 2015)
  - U.S. bank size distribution (Fernholz and Koch, 2015)

Summary and Conclusion

Three main contributions:

1. Application of new rank-based solution technique (Fernholz, 2015)
   - Nonparametric approach that can be used to solve many models
2. Detailed description and analysis of economic mobility
   - Analytic and numerical results for several measures of mobility
3. Examination of the implications of risk-sharing subgroups of households
   - Results about welfare and distributional implications

The End

Thank You