A Statistical Model of Inequality

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Income and Wealth Distributions

- Large and growing empirical literature focuses on measuring income and wealth distributions
  - Wolff (2010), Atkinson et al. (2011), Davies et al. (2011)
- Recent evidence suggests that income inequality, and possibly also wealth inequality, is growing in some places
  - Atkinson et al. (2011), Saez and Zucman (2014)
- Will these trends continue or reverse in the future?
- What are the causes?

Models of Income and Wealth Distributions

- Many empirical models of income and wealth processes, distributions
  - Guvenen (2009), Guvenen, Karahan, Ozkan, and Song (2014)
  - Browning et al. (2010), Altonji, Smith, and Vidangos (2013)
- Theoretical models of these distributions are also common
  - Benhabib, Bisin, and Zhu (2011, 2014), Fernholz (2014)
  - Aiyagari (1994), Cagetti and DeNardi (2008)
- All part of a broader literature on power laws in economics and finance
  - Gabaix (1999, 2009), Banner et al. (2005)

A Statistical Model of Inequality

- General model of rank-based systems applied to wealth distribution
  - Explicitly heterogeneous households subject to aggregate and idiosyncratic fluctuations in wealth holdings
  - In contrast to previous literature, almost no parametric structure on household behavior and the types of shocks that households face
- Closed-form characterization of the stable distribution of wealth
  - Describes wealth holdings for every rank in the distribution in terms of just two factors

Contributions

- Statistical model provides simple description of wealth distribution:

$$\text{inequality} = \frac{\text{idiosyncratic volatilities of wealth}}{\text{reversion rates of wealth}}$$

  - Volatilities, reversion rates vary across different ranked HHs

- Potential to understand how many different issues affect inequality
  - Institutions, policy, skill-biased technical change, globalization

- Model can be parameterized to exactly match any distribution
  - Construct such a parameterization for U.S. distribution of wealth
  - Detailed new wealth shares data from Saez and Zucman (2014)

U.S. Wealth Distribution

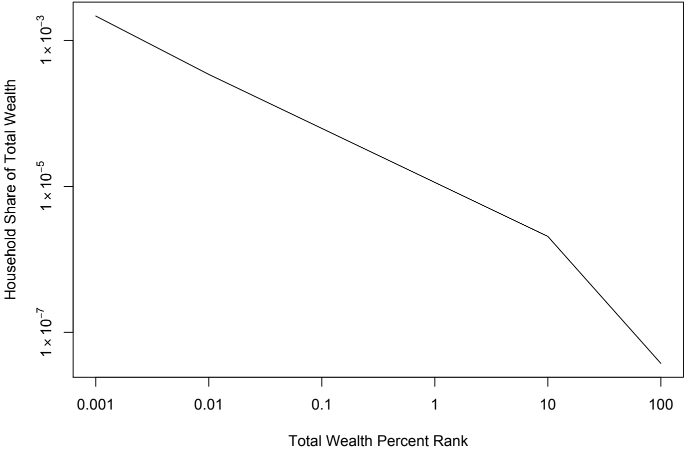

Figure: Household wealth shares for the model parameterized to match the U.S. wealth distribution in 2012.

Changing Wealth Shares and Progressive Capital Taxes

- Some recent data suggest an upward trend in top wealth shares
  - Distribution is transitioning, want to know where it is transitioning to
  - Use model to generate empirical estimates of future stable distribution
  - Saez & Zucman (2014): U.S. separating into divergent subpopulations?
- How would a progressive capital tax of 1-2% levied on 1% of households affect U.S. wealth distribution?
  - In principle, model can estimate distributional effect of any tax
  - In practice, hard to estimate this except, maybe, for a capital tax
  - If 1% tax reduces household's growth rate of wealth 1%, then big effect

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Basics

- Economy is populated by N households, time $t \in [0, \infty)$ is continuous
- Total wealth of each household given by process w_i:

$$d \log w_i(t) = \mu_i(t)\,dt + \sum_{z=1}^{M} \delta_{iz}(t)\,dB_z(t)$$

  - $B_1, \ldots, B_M$ are independent Brownian motions ($M \geq N$)
  - Little structure imposed on μ_i and δ_{iz}, more general than previous literature (Gabaix, 1999; Guvenen, 2009; Altonji et al., 2013)
  - Consistent with general equilibrium (Fernholz, 2014)
- Only for i.i.d.-like processes is model clearly inappropriate

Uninsurable Idiosyncratic Risk

The total wealth of each household $i = 1, \ldots, N$ is given by the process w_i:

$$d \log w_i(t) = \mu_i(t)\,dt + \sum_{z=1}^{M} \delta_{iz}(t)\,dB_z(t)$$

- A key assumption is that no two households' wealth dynamics are perfectly correlated over time
- In other words, households are subject to idiosyncratic fluctuations in their wealth holdings
  - Labor income: Aiyagari (1994), Krussel and Smith (1998)
  - Capital income: Angeletos and Calvet (2006), Benhabib et al. (2011)

Rank-Based Wealth Dynamics and Local Times

If $w_{(k)}(t)$ is the total wealth of the k-th wealthiest household, then the wealth dynamics follow

$$d\log w_{(k)}(t) = \mu_{p_t(k)}(t)\,dt + \sum_{z=1}^{M} \delta_{p_t(k)z}(t)\,dB_z(t) + \frac{1}{2}d\Lambda_{\log w_{(k)} - \log w_{(k+1)}}(t) - \frac{1}{2}d\Lambda_{\log w_{(k-1)} - \log w_{(k)}}(t).$$

- $p_t(k) = i$ when household i is the k-th wealthiest household
- Λ_x is the *local time* at 0 for the process x
  - Measures amount of time x spends near 0 (Karatzas and Shreve, 1991)

Relative Growth Rates and Volatilities

$$d \log w_{(k)}(t) = \mu_{p_t(k)}(t)\,dt + \sum_{z=1}^{M} \delta_{p_t(k)z}(t)\,dB_z(t) + \text{ local time terms}$$

Let α_k be the relative growth rate of the k-th wealthiest household,

$$\alpha_k = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\mu_{p_t(k)}(t) - \mu(t)\right)\,dt,$$

where $\mu(t)$ is growth rate of total wealth $w(t) = w_1(t) + \cdots + w_N(t)$.

Let σ_k be the volatility of relative wealth holdings,

$$\sigma_k^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{z=1}^{M} \left(\delta_{p_t(k)z}(t) - \delta_{p_t(k+1)z}(t)\right)^2\,dt.$$

Wealth Shares

Let $\theta_{(k)}(t)$ be share of total wealth held by k-th wealthiest household:

$$\theta_{(k)}(t) = \frac{w_{(k)}(t)}{w(t)}.$$

It is not hard to show that the relative wealth holdings of adjacent households in the distribution, $\log\theta_{(k)} - \log\theta_{(k+1)}$, satisfies

$$\begin{aligned} d\left(\log\theta_{(k)}(t) - \log\theta_{(k+1)}(t)\right) = {} & \left(\mu_{p_t(k)}(t) - \mu_{p_t(k+1)}(t)\right)\,dt \\ & - \frac{1}{2}d\Lambda_{\log\theta_{(k+1)}-\log\theta_{(k+2)}}(t) - \frac{1}{2}d\Lambda_{\log\theta_{(k-1)}-\log\theta_{(k)}}(t) \\ & + d\Lambda_{\log\theta_{(k)}-\log\theta_{(k+1)}}(t) + \sum_{z=1}^{M}\left(\delta_{p_t(k)z}(t) - \delta_{p_t(k+1)z}(t)\right)\,dB_z(t). \end{aligned}$$

Stable Version

The *stable version* of the process $\log \theta_{(k)} - \log \theta_{(k+1)}$ is defined by

$$d\left(\log \hat{\theta}_{(k)}(t) - \log \hat{\theta}_{(k+1)}(t)\right) = -\kappa_k\, dt + d\Lambda_{\log \hat{\theta}_{(k)} - \log \hat{\theta}_{(k+1)}}(t) + \sigma_k\, dB(t),$$

where

$$\kappa_k = \lim_{T\to\infty} \frac{1}{T} \Lambda_{\log \theta_{(k)} - \log \theta_{(k+1)}}(T).$$

The stable version uses time-averaged limits:

$$\begin{aligned} d\left(\log \theta_{(k)}(t) - \log \theta_{(k+1)}(t)\right) &= \left(\mu_{p_t(k)}(t) - \mu_{p_t(k+1)}(t)\right)\, dt \\ &\quad -\frac{1}{2} d\Lambda_{\log \theta_{(k+1)} - \log \theta_{(k+2)}}(t) - \frac{1}{2} d\Lambda_{\log \theta_{(k-1)} - \log \theta_{(k)}}(t) \\ &\quad + d\Lambda_{\log \theta_{(k)} - \log \theta_{(k+1)}}(t) + \sum_{z=1}^{M} \left(\delta_{p_t(k)z}(t) - \delta_{p_t(k+1)z}(t)\right)\, dB_z(t). \end{aligned}$$

Theorem (Distribution of Wealth)

There is a stable distribution of wealth in this economy if and only if $\alpha_1 + \cdots + \alpha_k < 0$, *for* $k = 1, \ldots, N-1$. *Furthermore, if there is a stable distribution of wealth, then for* $k = 1, \ldots, N-1$, *this distribution satisfies*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\log \hat{\theta}_{(k)}(t) - \log \hat{\theta}_{(k+1)}(t)\right) dt = \frac{\sigma_k^2}{-4(\alpha_1 + \cdots + \alpha_k)}.$$

- Stable distribution entirely determined by two factors
  1. Idiosyncratic volatility of wealth holdings: σ_k^2
  2. Reversion rates of wealth: $-\alpha_k$
- What is the effect of policy, institutions, or technology on inequality?
  - Must understand their effect on relative growth rates and volatilities

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- Without mean reversion condition $\alpha_1 + \cdots + \alpha_k < 0$, a subset of households will separate from the rest of the population
  - Top subset of households eventually forms its own stable distribution
  - This subset is fastest-growing subset of households in the economy

Stable Version?

- What is lost by considering stable versions of $\theta_{(k)}$?
  - Stable version replaces terms with their time-averaged limits

- Not much, as long as reversion rates, volatilities, and local times do not often change abruptly

$$\alpha_k = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\mu_{p_t(k)}(t) - \mu(t)\right)\, dt$$

$$\sigma_k^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{z=1}^{M} \left(\delta_{p_t(k)z}(t) - \delta_{p_t(k+1)z}(t)\right)^2\, dt$$

$$\kappa_k = \lim_{T\to\infty} \frac{1}{T} \Lambda_{\log\theta_{(k)} - \log\theta_{(k+1)}}(T)$$

The U.S. Wealth Distribution

Model can be parameterized to match any distribution:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\log \hat{\theta}_{(k)}(t) - \log \hat{\theta}_{(k+1)}(t)\right) dt = \frac{\sigma_k^2}{-4(\alpha_1 + \cdots + \alpha_k)}$$

Given data availability, choose to use estimates of $\theta_{(k)}$ and σ_k to infer α_k

- Set number of households $N = 1,000,000$
- Detailed new wealth shares data from Saez and Zucman (2014)
- Estimate σ_k, which measures volatility of $\log \theta_{(k)} - \log \theta_{(k+1)}$, the relative wealth holdings of adjacent households in the distribution

Volatility of Relative Wealth Holdings

The dynamics of household wealth over time:

$$dw_i(t) = w_i(t)\left(r_i(t) + \frac{\lambda_i(t) - c_i(t)}{w_i(t)}\right) dt$$

- Idiosyncratic investment returns
  - Ownership of primary housing, private equity
  - Std. deviation of 0.2 (Flavin & Yamashita, 2002; Angeletos, 2007)
- Idiosyncratic fluct. in labor income minus consumption rel. to wealth
  - Std. deviation of labor income of 0.5 (Guvenen et al., 2014)
  - Combine with SCF earnings, wealth data (Diaz-Gimenez et al., 2011)
  - Assume idiosyncratic change in $\lambda_i(t)$ is change in $\lambda_i(t) - c_i(t)$

Volatility Estimates

- Some uncertainty surrounding these estimates of σ_k
- Empirical work suggests true values between low and high estimates

Household Wealth Percentile	Low Estimate Volatility σ_k	High Estimate Volatility σ_k
0-10	0.283	0.286
10-20	0.283	0.294
20-40	0.283	0.316
40-60	0.283	0.392
60-100	0.283	1.662

Wealth Shares

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\log \hat{\theta}_{(k)}(t) - \log \hat{\theta}_{(k+1)}(t)\right) dt = \frac{\sigma_k^2}{-4(\alpha_1 + \cdots + \alpha_k)}$$

- What are the correct values for household wealth shares $\theta_{(k)}$?
  - Observe wealth shares for some groups, but not every single $\theta_{(k)}$
- Assume Pareto-like distribution with varying parameter
  - More general than standard Pareto, matches the data better
  - Varying parameter across just 3 groups achieves nearly perfect match
  - Once $\theta_{(k)}$ are set, rank-based reversion rates $-\alpha_k$ can be inferred

2012 U.S. Wealth Distribution

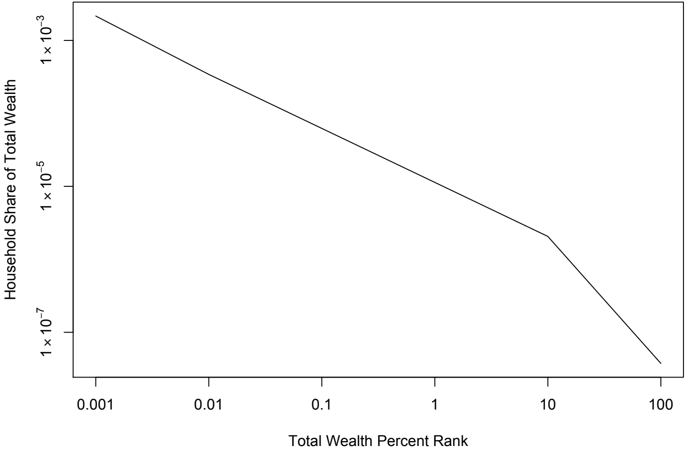

Figure: Household wealth shares for the model parameterized to match the U.S. wealth distribution in 2012.

Trends in U.S. Wealth Shares

- Parameterization process works for any empirical distribution
  - Important requirement is that the distribution is stable
  - Wealth shares should not be trending up or down
- Far from clear that U.S. wealth distribution is currently stable
  - Saez and Zucman (2014), SCF data
- Stability issues can be addressed using this methodology
  - Where is the distribution transitioning to?
  - Estimate the future stable distribution of wealth by appropriately adjusting the rank-based reversion rates $-\alpha_k$

Top U.S. Wealth Shares over Time

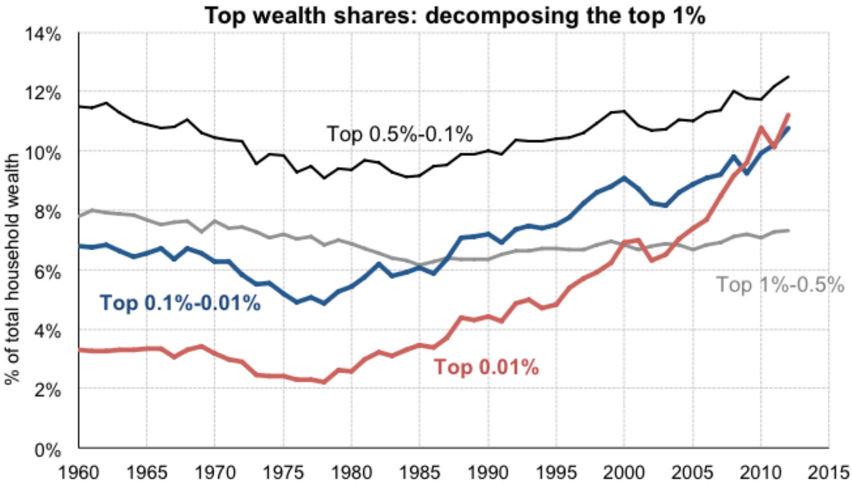

Figure: Top U.S. wealth shares 1960-2012, from Saez and Zucman (2014).

The Trajectory of the U.S. Distribution of Wealth

- What are the appropriate adjustments for the relative growth rates?
  - Add rate at which wealth shares change to corresponding values of α_k
  - If stable, reversion rates are $-\alpha_k$; since unstable, must adjust α_k by observed rate of changing wealth shares
- Consider 4 different scenarios for changing wealth shares:
  1. 2012 U.S. wealth distribution is stable
  2. Top 0.01% increasing by 0.5% per year
  3. Top 0.01% and 0.1-0.01% increasing by 1.5% and 0.5% per year, bottom 90% decreasing by 0.5% per year
  4. Top 0.01%, 0.1-0.01%, and 0.5-0.1% increasing by 2.5%, 1.5%, and 0.5% per year, bottom 90% decreasing by 1.5% per year

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Four Scenarios for the Future U.S. Wealth Distribution

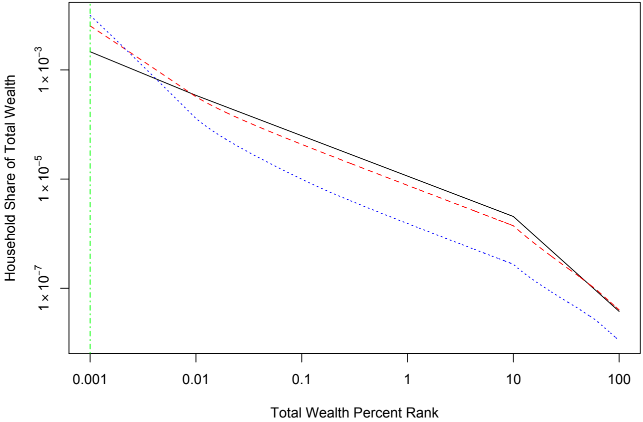

Figure: Household wealth shares for high estimates of the volatilities σ_k under Scenarios 1 (solid black line), 2 (dashed red line), 3 (dotted blue line), and 4 (vertical dot-dashed green line).

A Divergent Trajectory?

- These are not precise forecasts of the future, but rather estimates of the current trajectory in the absence of future changes
  - If economic environment changes, then so will the trajectory
- In fact, some data point to rapidly increasing top shares that are difficult to reconcile with any stable distribution
  - Saez and Zucman (2014) vs. adjusted SCF vs. unadjusted SCF
  - Stability requires $\alpha_1 + \cdots + \alpha_k < 0$, for all $k = 1, \ldots, N-1$
- According to Saez & Zucman (2014) data, U.S. distribution might be temporarily unstable and separating into divergent subpopulations
  - Suggests that some aspect of economic environment will likely change

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  - Suggests that some aspect of economic environment will likely change

The Distributional Implications of Taxes

- In principle, model can estimate distributional effects of any tax
  - Just estimate how tax affects values of α_k and σ_k
- In practice, hard to measure this except, maybe, for a capital tax
- Assume 1% tax reduces taxed HH's growth rate of wealth by 1%
  - This ignores incentive effects of taxes
  - Useful as baseline starting case (other effects can then be incorporated)
  - In terms of the model, 1% tax reduces α_k by 0.01

A Progressive Capital Tax

- Consider a simple progressive capital tax
  - Top 0.5% of HHs pay rate of 2%, top 0.5-1% of HHs pay rate of 1%
  - All other households pay nothing
- This is similar to the tax proposed by Piketty (2014) for Europe
  - Piketty (2014) does not estimate distributional effects of his tax, only the effect on government revenues
  - Discussion has focused on distortions vs. revenues
  - No quantitative estimates of the distributional implications so far

Progressive Capital Tax: Scenario 1

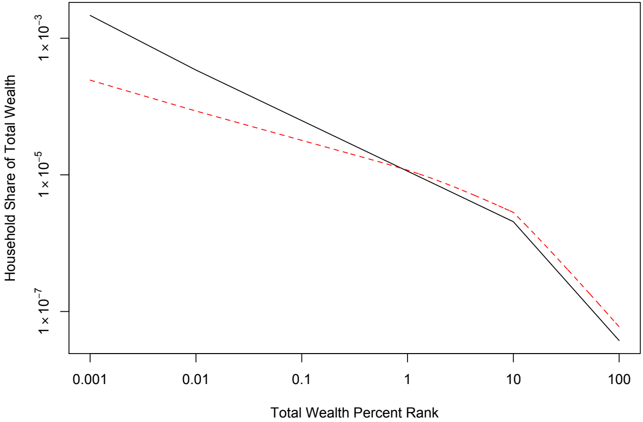

Figure: Household wealth shares with (dashed red line) and without (solid black line) a 1-2% progressive capital tax on the top 1% of households for high estimates of the volatilities σ_k under Scenario 1.

Progressive Capital Tax: Scenario 2

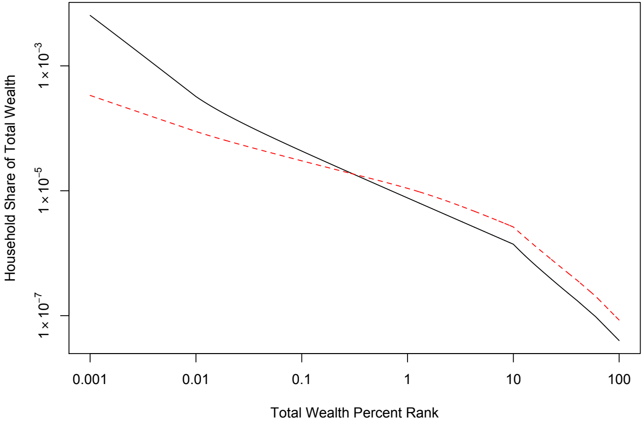

Figure: Household wealth shares with (dashed red line) and without (solid black line) a 1-2% progressive capital tax on the top 1% of households for high estimates of the volatilities σ_k under Scenario 2.

Progressive Capital Tax: Scenario 3

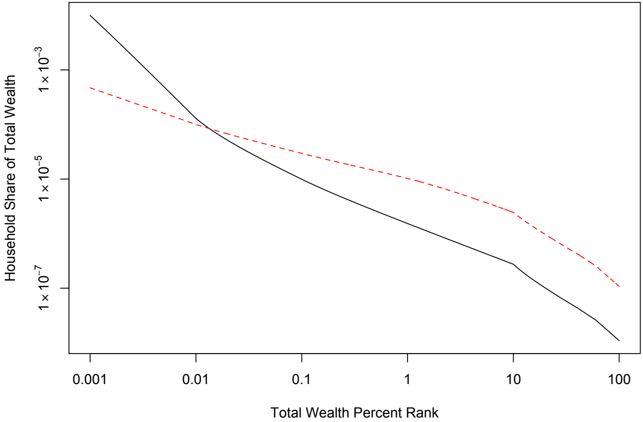

Figure: Household wealth shares with (dashed red line) and without (solid black line) a 1-2% progressive capital tax on the top 1% of households for high estimates of the volatilities σ_k under Scenario 3.

Progressive Capital Tax: Scenario 4

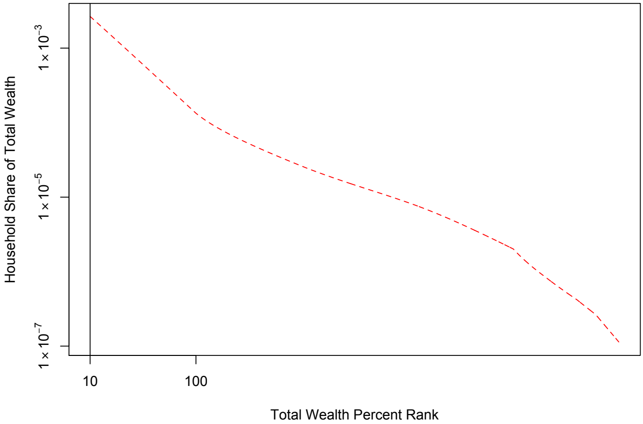

Figure: Household wealth shares with (dashed red line) and without (vertical solid black line) a 1-2% progressive capital tax on the top 1% of households for high estimates of the volatilities σ_k under Scenario 4.

Progressive Capital Tax: Scenario 1

Household Wealth Percentile	No Capital Tax	Low Estimate Volatility σ_k	High Estimate Volatility σ_k
0-0.01	11.0%	1.4%	1.4%
0.01-0.1	10.1%	3.8%	3.9%
0.1-0.5	11.9%	7.9%	8.1%
0.5-1	6.9%	6.4%	6.5%
1-10	35.1%	43.4%	43.5%
10-100	25.1%	37.1%	36.6%

Table: Household wealth shares with a 1-2% progressive capital tax on the top 1% of households for different estimates of the volatilities σ_k under Scenario 1.

The Effect of a Progressive Capital Tax

- Progressive capital tax of 1-2% on 1% of households substantially reshapes the distribution of wealth and reduces inequality
  - If 2012 U.S. wealth distribution is stable, then inequality reduced to levels similar to 1970s U.S.
- What is the intuition for this large effect?
  - Top 1% hold 40% of total wealth, so tax affects 40% of economy
- Results are definitely not an endorsement of this policy
  - No welfare or cost-benefit analysis

Extensions and Applications

- A statistical model
  - Model can be applied to rank-based systems other than wealth
  - Clearly inappropriate only for unstable or i.i.d.-like processes
- Some possible applications
  - World income distribution: Are we converging, and if so, to what?
  - City size: Like Gabaix (1999), but without ex-ante identical cities
  - Income: Is it possible to improve on standard, AR1-style approach?
- General approach can be used in theoretical models, too
  - Fernholz (2015) does this for wealth, but may be applicable elsewhere

Recap

- A statistical model of inequality
  - Few restrictions on household wealth processes

- Closed-form characterization of the stable distribution of wealth:

$$\text{inequality} = \frac{\text{idiosyncratic volatilities of wealth}}{\text{reversion rates of wealth}}$$

  - Potential to understand how many different issues affect inequality

- A parameterization of the U.S. distribution of wealth
  - Future distribution quite sensitive to underlying trends in top shares
  - Small progressive capital tax substantially reshapes distribution

The End

Thank You