Empirical Methods for Dynamic Power Law Distributions in the Social Sciences

Ricardo T. Fernholz

Claremont McKenna College

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Power Law Distributions

- Power laws are characterized by a linear relationship between log size and log rank
- Power laws are common in economics, finance, and the social sciences more broadly
  - Income and wealth: Atkinson, Piketty and Saez (2011), Piketty (2014)
  - Firm size: Simon and Bonini (1958), Luttmer (2007, 2011)
  - Bank size: Janicki and Prescott (2006), Fernholz and Koch (2016)
  - City size: Gabaix (1999), Ioannides and Skouras (2013)

Random Growth Processes and Power Laws

- Power laws and Pareto distributions commonly modeled as the result of random growth processes
  - Champernowne (1953), Luttmer (2007), Benhabib, Bisin & Zhu (2011)
- Random growth following Gibrat's law in the presence of some friction yields a power law distribution
  - Gabaix (1999) uses this basic insight to generate Zipf's law for cities
  - Many papers use this basic insight to generate power laws

Empirical Methods for Dynamic Power Law Distributions

- Rank-based, nonparametric methods characterize general power law distributions in any continuous random growth setting
  - Unifying framework that encompasses and extends previous literature
  - Up to now, no empirical methods for dynamic power laws in economics

- Provides simple description of stationary distribution:

$$\text{concentration} = \frac{\text{idiosyncratic volatilities}}{\text{reversion rates}}$$

  - Reversion rates measure cross-sectional mean reversion

Applications

- Growing concentration of U.S. banking assets starting in the 1990s
  - Fernholz and Koch (2016)
- The distribution of relative commodity prices
  - Methods accurately describe distribution of relative commodity prices
  - Future commodity price predictability based on rank
- Many other potential applications in economics and finance
  - Increasing inequality (Atkinson et al., 2011; Saez and Zucman, 2014)
  - Increasing house price dispersion (Van Nieuwerburgh and Weill, 2010)

Basics

- Economy is populated by N agents, time $t \in [0, \infty)$ is continuous
- Total unit holdings of each agent given by process x_i:

$$d \log x_i(t) = \mu_i(t)\, dt + \sum_{s=1}^{M} \delta_{is}(t)\, dB_s(t)$$

  - $B_1, \ldots, B_M$ are independent Brownian motions $(M \geq N)$
  - Nonparametric approach with little structure imposed on μ_i and δ_{is}
  - More general than previous random growth literature based on equal growth rates and volatilities of Gibrat's Law (Gabaix, 1999, 2009)

Rank-Based Unit Dynamics and Local Times

Let $x_{(k)}(t)$ be the unit holdings of the k-th ranked agent:

$$d\log x_{(k)}(t) = \mu_{p_t(k)}(t)\,dt + \sum_{s=1}^{M} \delta_{p_t(k)s}(t)\,dB_z(t) + \frac{1}{2}d\Lambda_{\log x_{(k)} - \log x_{(k+1)}}(t) - \frac{1}{2}d\Lambda_{\log x_{(k-1)} - \log x_{(k)}}(t)$$

- $p_t(k) = i$ when agent i has k-th largest unit holdings
- Λ_z is the *local time* at 0 for the process z
  - Measures amount of time z spends near 0 (Karatzas and Shreve, 1991)

Let $\theta_{(k)}(t)$ be share of total units held by k-th ranked agent:

$$\theta_{(k)}(t) = \frac{x_{(k)}(t)}{x(t)} = \frac{x_{(k)}(t)}{x_1(t) + \cdots + x_N(t)}$$

Rank-Based Unit Dynamics and Local Times

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Relative Growth Rates and Volatilities

$$d \log x_{(k)}(t) = \mu_{p_t(k)}(t)\, dt + \sum_{s=1}^{M} \delta_{p_t(k)s}(t)\, dB_s(t) \; + \text{ local time terms}$$

Let α_k be the relative growth rate of the k-th ranked agent,

$$\alpha_k = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\mu_{p_t(k)}(t) - \mu(t)\right)\, dt,$$

where $\mu(t)$ is growth rate of total units $x(t) = x_1(t) + \cdots + x_N(t)$.

Let σ_k be the volatility of relative unit holdings $\log \theta_{(k)} - \log \theta_{(k+1)}$,

$$\sigma_k^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{s=1}^{M} \left(\delta_{p_t(k)s}(t) - \delta_{p_t(k+1)s}(t)\right)^2\, dt.$$

Reversion Rates and Idiosyncratic Volatilities

- Refer to $-\alpha_k$ as reversion rates of unit holdings
  - Equal to minus the growth rate of units for the rank k agent relative to the growth rate of total units of all agents
  - A measure of cross-sectional mean reversion

- Parameters σ_k measure idiosyncratic unit volatility
  - Measures volatility of relative unit holdings of adjacent ranked agents
  - This includes shocks that affect only one agent as well as shocks that affect multiple agents in different ways

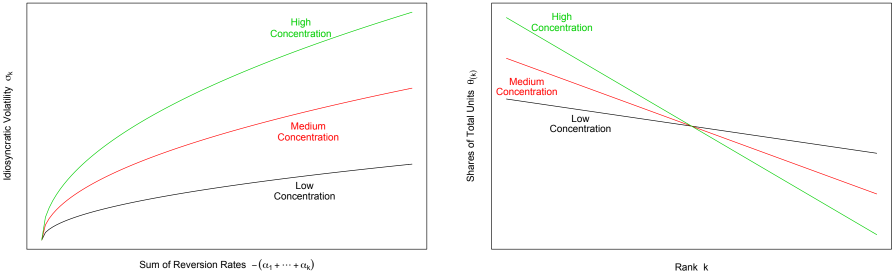
Theorem

There is a stationary distribution of unit holdings by agents if and only if $\alpha_1 + \cdots + \alpha_k < 0$*, for* $k = 1, \ldots, N-1$*. Furthermore, if there is a stationary distribution, then for* $k = 1, \ldots, N-1$*, this distribution satisfies*

$$E\left[\log \theta^*_{(k)}(t) - \log \theta^*_{(k+1)}(t)\right] = \frac{\sigma_k^2}{-4(\alpha_1 + \cdots + \alpha_k)}.$$

- Distribution shaped entirely by two factors
  1. Idiosyncratic unit volatilities: σ_k
  2. Reversion rates of unit holdings: $-\alpha_k$
- Only a change in these factors can alter the distribution
- Theorem describes behavior of stable versions of unit shares, $\theta^*_{(k)}$

Idiosyncratic Volatility, Reversion Rates, and Concentration



$$E\left[\log \theta^*_{(k)}(t) - \log \theta^*_{(k+1)}(t)\right] = \frac{\sigma_k^2}{-4(\alpha_1 + \cdots + \alpha_k)}$$

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\theta_{(k)}(t) & \longrightarrow & \theta_{(k)}(t+1) \\
\vdots & & \vdots \\
\vdots & & \vdots
\end{array}
$$

$$
\begin{tikzcd}
\vdots & \vdots \\
\theta_{p_t(k)}(t) = \theta_{(k)}(t) \arrow[r, bend left] \arrow[dr] & \theta_{p_t(j)}(t+1) = \theta_{(k)}(t+1) \\
\vdots & \theta_{p_t(k)}(t+1) \\
\theta_{p_t(j)}(t) = \theta_{(j)}(t) \arrow[uur] & \vdots \\
\vdots & \vdots
\end{tikzcd}
$$

Mean-Reversion Condition

Theorem

There is a stationary distribution of unit holdings by agents if and only if $\alpha_1 + \cdots + \alpha_k < 0$, for $k = 1, \ldots, N-1$. Furthermore, if there is a stationary distribution, then for $k = 1, \ldots, N-1$, this distribution satisfies

$$E\left[\log \theta^*_{(k)}(t) - \log \theta^*_{(k+1)}(t)\right] = \frac{\sigma_k^2}{-4(\alpha_1 + \cdots + \alpha_k)}.$$

- Unit holdings of top k agents must on average grow more slowly than unit holdings of bottom $N-k$ agents
  - Otherwise, the distribution of unit holdings is asymptotically degenerate
- There is a rank-based predictability for agents' future unit holdings

Top k Agents at time t		Top k Agents at time $t+1$	
	$\left[\begin{array}{c} \theta_{(1)}(t) \\ \theta_{(2)}(t) \\ \vdots \\ \theta_{(k)}(t) \end{array}\right.$		$\left[\begin{array}{c} \theta_{(1)}(t+1) \\ \theta_{(2)}(t+1) \\ \vdots \\ \theta_{(k)}(t+1) \end{array}\right.$
	$\left[\begin{array}{c} \theta_{(k+1)}(t) \\ \theta_{(k+2)}(t) \\ \vdots \\ \theta_{(N)}(t) \end{array}\right.$		$\left[\begin{array}{c} \theta_{(k+1)}(t+1) \\ \theta_{(k+2)}(t+1) \\ \vdots \\ \theta_{(N)}(t+1) \end{array}\right.$

Relation to Previous Literature

- Rank-based, nonparametric approach nests much of previous literature
- Gibrat's law: Growth rates and volatilities equal for all agents
  - Gabaix (2009) shows that Gibrat's law yields a Pareto distribution
  - Gabaix (1999) shows that Gibrat's law sometimes yields Zipf's law
- Gibrat's law: $\alpha = \alpha_1 = \cdots = \alpha_{N-1}$ and $\sigma = \sigma_1 = \cdots = \sigma_{N-1}$
- This implies that:

$$E\left[\log\theta^*_{(k)}(t) - \log\theta^*_{(k+1)}(t)\right] = \frac{\sigma_k^2}{-4(\alpha_1 + \cdots + \alpha_k)} = \frac{\sigma^2}{-4k\alpha}$$

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Gibrat's Law and Pareto Distributions

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- Gibrat's law: $\alpha = \alpha_1 = \cdots = \alpha_{N-1}$ and $\sigma = \sigma_1 = \cdots = \sigma_{N-1}$
- Log-log plot of shares $\theta_{(k)}$ vs. rank k has constant slope (Pareto):

$$\frac{E\left[\log \theta^*_{(k)}(t) - \log \theta^*_{(k+1)}(t)\right]}{\log k - \log k + 1} \approx \frac{-k\sigma^2}{-4k\alpha} = \frac{\sigma^2}{4\alpha}$$

Gibrat's Law and Zipf's Law

- Rank-based, nonparametric approach nests much of previous literature
- Gibrat's law: Growth rates and volatilities equal for all agents
  - Gabaix (2009) shows that Gibrat's law yields a Pareto distribution
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- Gibrat's law: $\alpha = \alpha_1 = \cdots = \alpha_{N-1}$ and $\sigma = \sigma_1 = \cdots = \sigma_{N-1}$
- Log-log plot of shares $\theta_{(k)}$ vs. rank k has slope -1 (Zipf's law):

$$\frac{E\left[\log\theta^*_{(k)}(t) - \log\theta^*_{(k+1)}(t)\right]}{\log k - \log k+1} \approx \frac{-k\sigma^2}{-4k\alpha} = \frac{\sigma^2}{4\alpha} = -1 \quad \text{iff} \quad \sigma^2 = -4\alpha$$

Estimation: Reversion Rates

It can be shown that for all $k = 1, \ldots, N-1$, the estimators $-\hat{\alpha}_k$ are increasing in the quantity

$$\log\left[\theta_{p_{t+1}(1)}(t+1) + \cdots + \theta_{p_{t+1}(k)}(t+1)\right] \\ - \log\left[\theta_{p_t(1)}(t+1) + \cdots + \theta_{p_t(k)}(t+1)\right].$$

Reversion rates measure the intensity of mean reversion, since they are increasing in the difference between the time $t+1$ units of the largest agents at $t+1$ and the time $t+1$ units of the largest agents at t.

Estimation: Idiosyncratic Volatilities

- Idiosyncratic volatilities measure variance of relative unit holdings for adjacent ranked agents, $\log\theta_{(k)} - \log\theta_{(k+1)}$
- Discrete-time approximation yields

$$\hat{\sigma}_k^2 = \frac{1}{T}\sum_{t=1}^{T}\Big[\big(\log\theta_{p_t(k)}(t+1) - \log\theta_{p_t(k+1)}(t+1)\big) \\ - \big(\log\theta_{p_t(k)}(t) - \log\theta_{p_t(k+1)}(t)\big)\Big]^2$$

Banking Assets in the U.S.

- Growing concentration of U.S. banking assets starting in the 1990s
- Fernholz and Koch (2016a): Why are Big Banks Getting Bigger?
  - Estimate volatilities and reversion rates using U.S. banking assets data
  - Stationary distribution for banking assets described by:

$$\text{asset concentration} = \frac{\text{idiosyncratic asset volatilities}}{\text{reversion rates of assets}}$$

- By estimating volatilities and reversion rates, can determine why, in an econometric sense, big banks got bigger

The Changing U.S. Bank Size Distribution

Figure: The share of total assets held by the largest U.S. bank-holding companies.

Idiosyncratic Volatilities

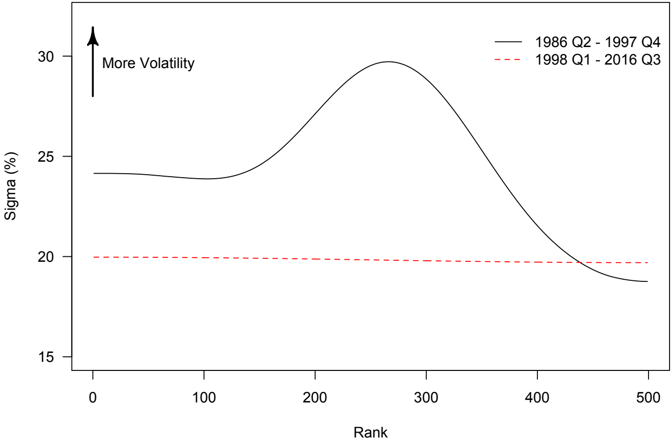


Figure: Standard deviations of idiosyncratic asset volatilities (σ_k) for different ranked bank-holding companies.

Reversion Rates

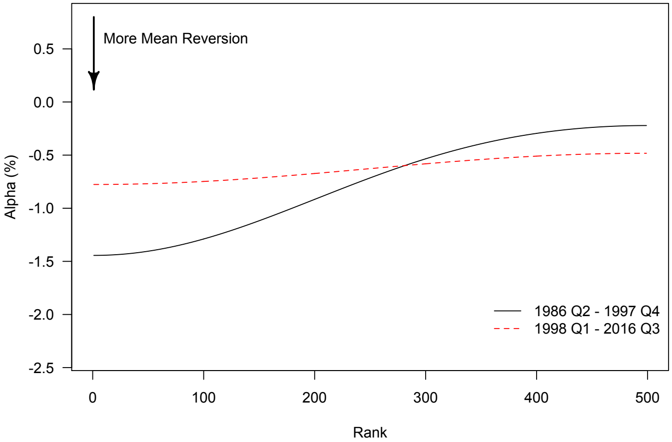
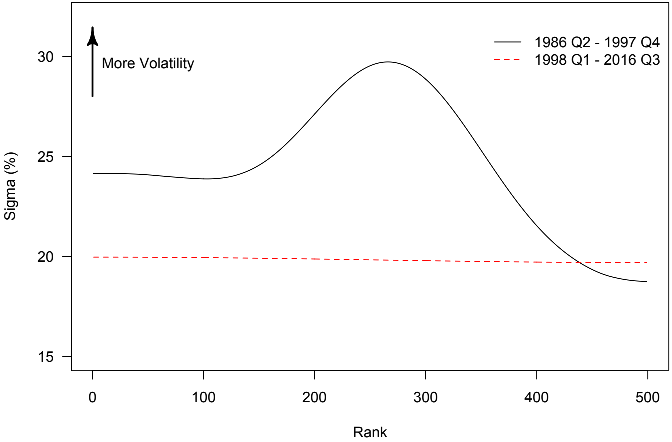


Figure: Minus the reversion rates (α_k) for different ranked bank-holding companies.

Idiosyncratic Volatilities: Beyond Gibrat's Law



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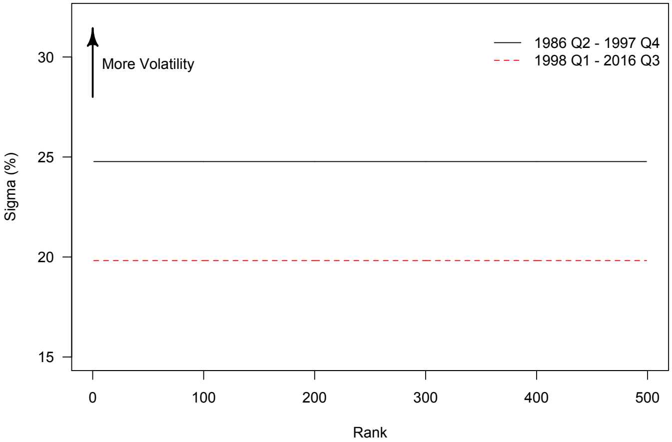


Figure: Standard deviations of idiosyncratic asset volatilities (σ_k) when imposing Gibrat's Law.

Prediction vs. Data: Beyond Gibrat's Law

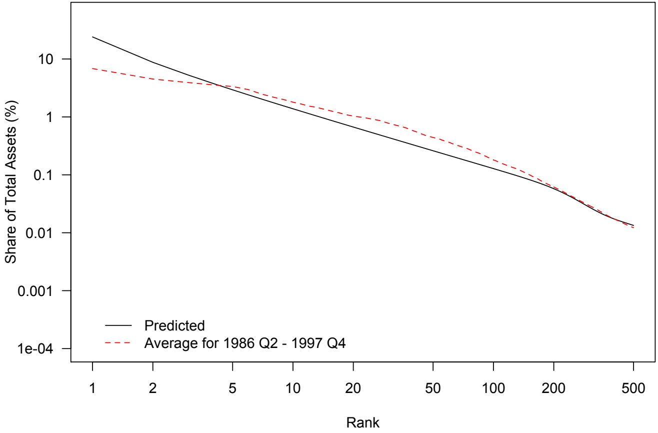
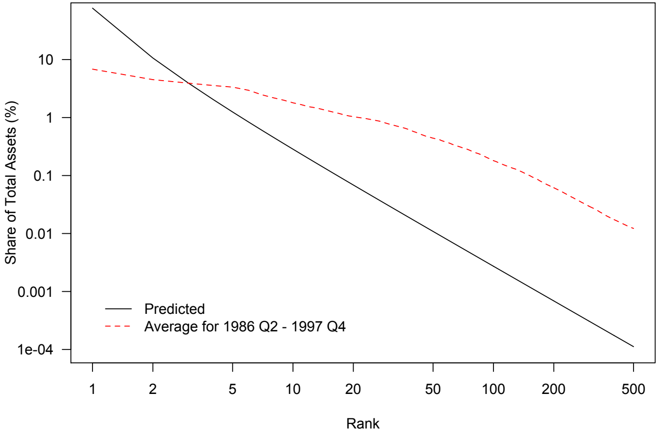


Figure: Shares of total assets held by the 500 largest U.S. BHCs for 1986 Q2 - 1997 Q4 as compared to the predicted shares.

Prediction vs. Data: Beyond Gibrat's Law



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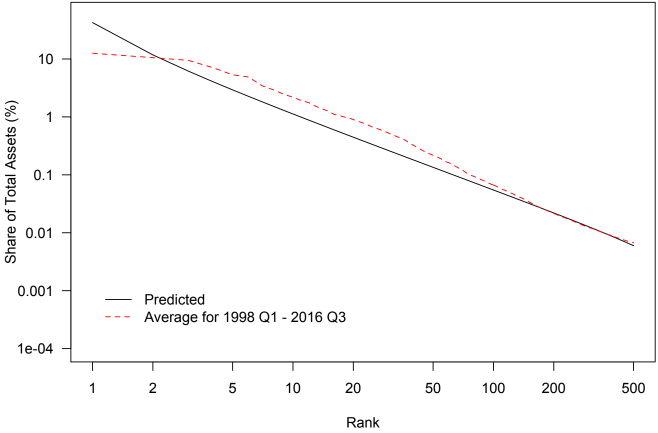

Figure: Shares of total assets held by the 500 largest U.S. BHCs for 1998 Q1 - 2016 Q3 as compared to the predicted shares.

Prediction vs. Data: Beyond Gibrat's Law

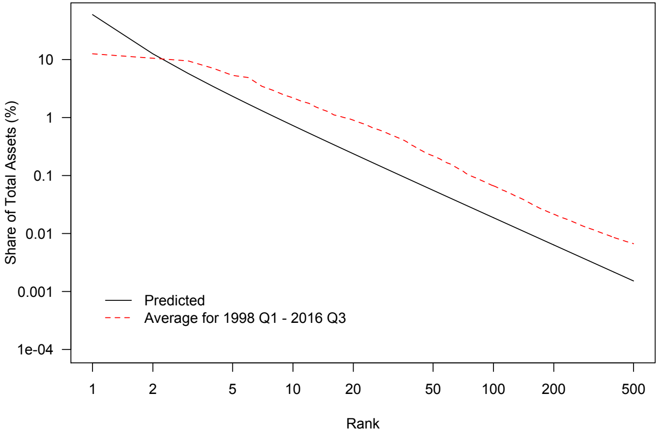

Figure: Shares of total assets held by the 500 largest U.S. BHCs for 1998 Q1 - 2016 Q3 as compared to the predicted shares when imposing Gibrat's Law.

Commodity Prices

- Results can also be applied to distribution of relative commodity prices
  - Instead of unit holdings by agents, let processes x_i represent prices of different commodities
  - Normalize prices by equalizing them in initial period, this way they can be compared in an economically meaningful way
- Let $\tilde{x}_{(k)}$ be k-th most expensive commodity price relative to average
  - Can show that $\log \tilde{x}_{(k)}(t) - \log \tilde{x}_{(k+1)}(t) = \log \theta_{(k)}(t) - \log \theta_{(k+1)}(t)$, so distributions of relative commodity prices and "price shares" are same
- Estimate volatility and reversion rates using commodity prices data
  - Monthly data for 22 common commodities obtained from FRED

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Relative Commodity Prices

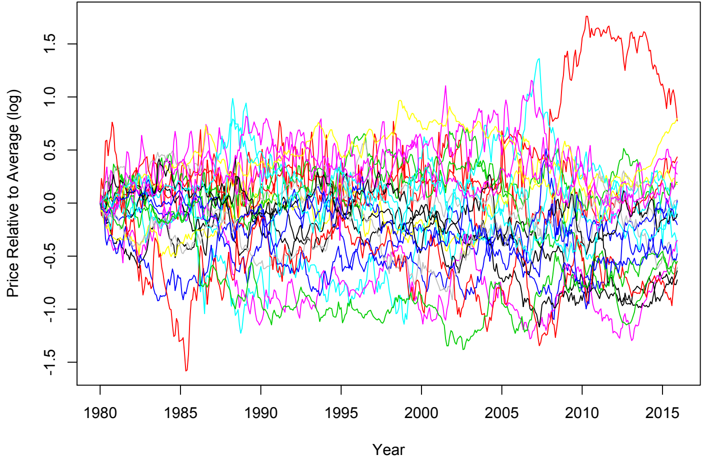

Figure: Commodity prices relative to the average, 1980 - 2015.

Relative Commodity Prices by Rank

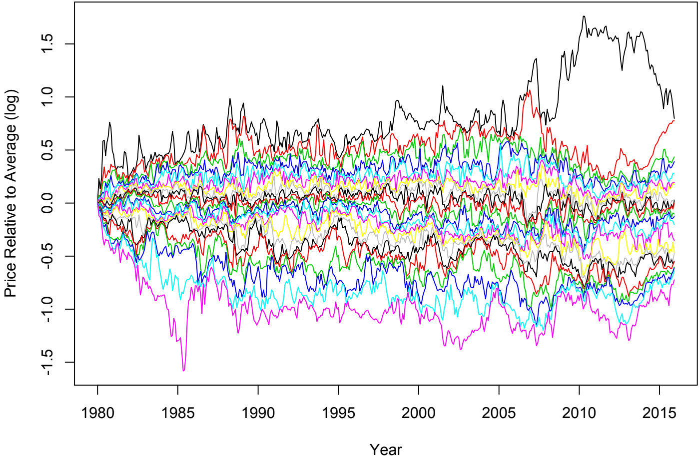

Figure: Commodity prices by rank relative to the average, 1980 - 2015.

Confidence Intervals and Parameter Smoothing

- Smooth estimated parameters α_k and σ_k across ranks using a Gaussian kernal smoother
  - Choose number of smoothings that minimizes the squared deviation between prediction and data

- Underlying distribution of parameters α_k and σ_k is unknown
- Bootstrap resampling generates confidence intervals
  - 10,000 replicate samples randomly generated with replacement
  - Samples consist of $T-1$ pairs of adjacent monthly prices
  - Confidence intervals based on range of estimates in these resamples

Reversion Rates

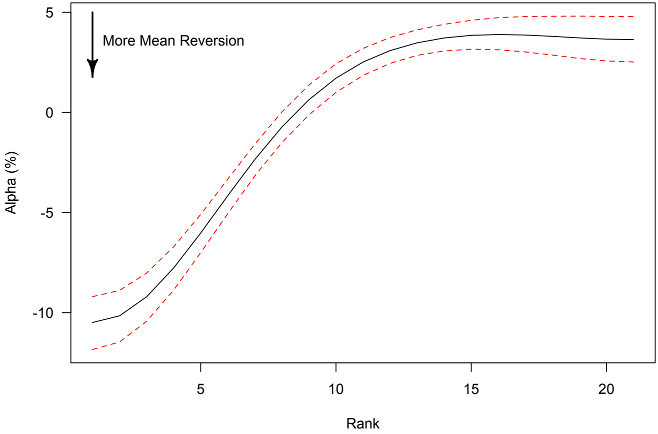

Figure: Point estimates and 95% confidence intervals of minus the reversion rates (α_k) for different ranked commodities, 1980 - 2015.

Idiosyncratic Volatilities

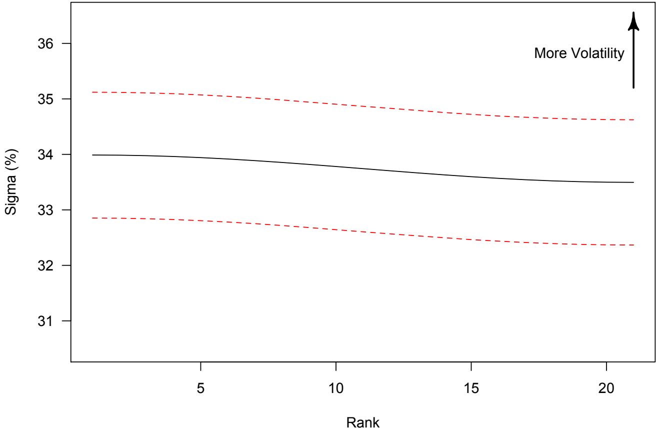

Figure: Point estimates and 95% confidence intervals of idiosyncratic commodity price volatilities (σ_k) for different ranked commodities, 1980 - 2015.

Prediction vs. Data

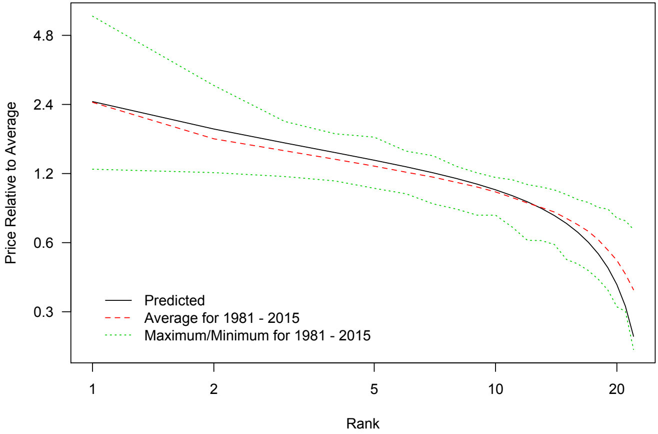

Figure: Relative commodity prices for different ranked commodities for 1981 - 2015 as compared to the predicted relative prices.

Cross-Sectional Mean Reversion Condition

Theorem

There is a stationary distribution of relative commodity prices if and only if $\alpha_1 + \cdots + \alpha_k < 0$*, for* $k = 1, \ldots, N-1$*. Furthermore, if there is a stationary distribution, then for* $k = 1, \ldots, N-1$*, this distribution satisfies*

$$E\left[\log \theta^*_{(k)}(t) - \log \theta^*_{(k+1)}(t)\right] = \frac{\sigma_k^2}{-4(\alpha_1 + \cdots + \alpha_k)}.$$

- The top k most expensive commodity prices must on average grow more slowly than the bottom $N-k$ commodity prices
  - Otherwise, commodity price distribution is asymptotically degenerate
- This is a testable prediction of these econometric methods

Rank Cutoff k	Average Difference	Standard Deviation	t-Statistic
1	-1.46%	9.34%	-3.24
2	-1.69%	7.06%	-4.96
3	-1.42%	5.89%	-4.98
4	-1.49%	4.98%	-6.19
5	-1.28%	4.57%	-5.80
6	-1.14%	4.20%	-5.62
7	-1.05%	3.93%	-5.51
8	-0.87%	3.72%	-4.83
9	-0.75%	3.45%	-4.47
10	-0.68%	3.45%	-4.07
11	-0.64%	3.40%	-3.89
12	-0.61%	3.37%	-3.73
13	-0.62%	3.29%	-3.90
14	-0.59%	3.37%	-3.59
15	-0.52%	3.41%	-3.17
16	-0.47%	3.53%	-2.77
17	-0.45%	3.82%	-2.41
18	-0.36%	3.80%	-1.98
19	-0.47%	4.22%	-2.32
20	-0.53%	4.87%	-2.23
21	-0.88%	7.16%	-2.55

Table: Difference between monthly log growth rates for top k ranked commodities minus bottom $N-k$ ranked commodities from 1980 - 2015.

Rank-Based Forecasts of Future Commodity Prices

Local-time-based estimation procedure yields a simple equation for rank-based forecasts of future commodity prices:

$$E_t\left[d\log\left(\tilde{x}_{p_t(1)}(t)+\cdots+\tilde{x}_{p_t(k)}(t)\right)\right]= -\frac{\tilde{x}_{(k)}(t)}{2(\tilde{x}_{(1)}(t)+\cdots+\tilde{x}_{(k)}(t))}E_t\left[d\Lambda_{\log\tilde{x}_{(k)}-\log\tilde{x}_{(k+1)}}(t)\right]$$

Use estimates of $d\Lambda_{\log\tilde{x}_{(k)}-\log\tilde{x}_{(k+1)}}$ to forecast change in top k relative commodity prices out of sample

1. Estimate $d\Lambda_{\log\tilde{x}_{(k)}-\log\tilde{x}_{(k+1)}}$ using fixed window (first 10 years)
2. Estimate $d\Lambda_{\log\tilde{x}_{(k)}-\log\tilde{x}_{(k+1)}}$ using rolling window (all previous months)

Local Time Processes

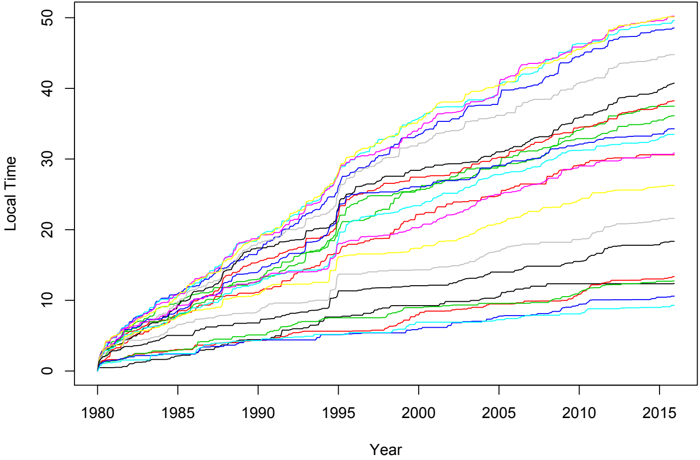

Figure: Local time processes for different ranked commodities, 1980 - 2015.

Rank Cutoff k	Fixed Estimation of κ_k	Rolling Estimation of κ_k
1	0.988	0.988
2	0.972	0.973
3	0.971	0.972
4	0.958	0.959
5	0.963	0.963
6	0.964	0.964
7	0.966	0.965
8	0.982	0.978
9	0.993	0.989
10	0.990	0.986
11	0.983	0.980
12	0.986	0.983
13	0.981	0.978
14	0.989	0.987
15	0.990	0.988
16	0.987	0.985
17	0.990	0.988
18	0.996	0.992
19	0.994	0.990
20	0.994	0.991
21	0.990	0.987

Table: RMSE ratios of one-month-ahead out-of-sample forecasts of log price of top k ranked commodities relative to price of all N commodities for 1990 - 2015.

Cross-Sectional Mean Reversion in Commodity Futures

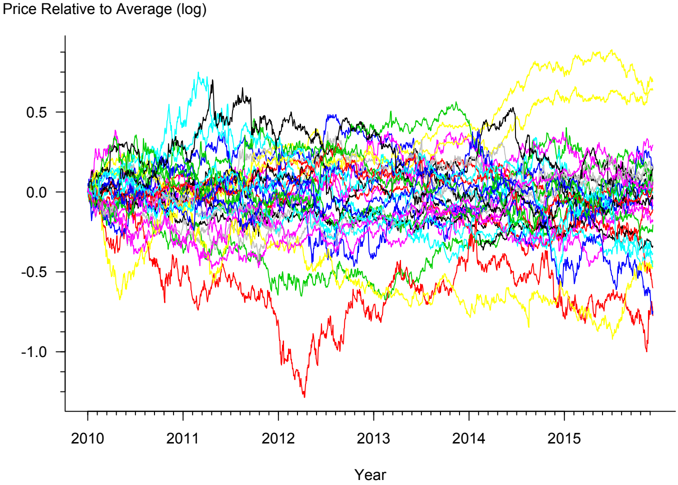

Figure: Commodity futures prices relative to the average, 2010 - 2016.

The Rank Effect

- Mean-reversion condition implies there is a rank effect for commodities (Fernholz and Koch, 2016b)
  - Higher-priced, higher-ranked commodities should grow more slowly (lower returns) than lower-priced, lower-ranked commodities
  - "Value" for commodities (Asness, Moskowitz, and Pedersen, 2013)

- Test for the rank effect using commodity futures data for 2010 - 2016
  - Portfolios are rebalanced every day and place equal weight on each commodity that goes into the portfolio
  - Prices are normalized to all equal each other on first day, then wait 20 days before forming portfolios so that rank has meaning

Returns: Low-Rank vs. High-Rank Commodities

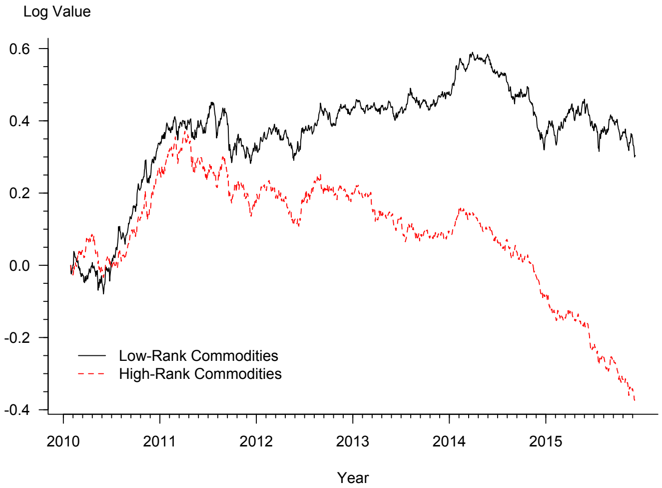

Figure: Log returns for low- and high-ranked commodities portfolios, 2010 - 2016.

Relative Returns: Low-Rank vs. High-Rank Commodities

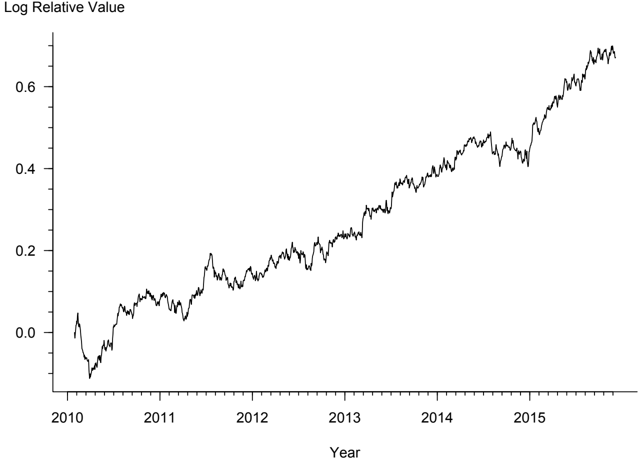

Figure: Log returns of low-rank commodities portfolio relative to high-rank commodities portfolio, 2010 - 2016.

Relative Returns: Low-Rank vs. High-Rank Commodities

- Lower-priced, lower-ranked commodities portfolio (bottom quintile) consistently outperforms higher-priced, higher-ranked commodities portfolio (top quintile)
  - Average yearly excess return of 23.2%
  - Sharpe ratio almost twice Russell 3000
  - Correlation with Russell 3000 returns (beta) of 0.10
  - Results are similar for other low-minus-high portfolio sorts such as median or decile

Equities, Bonds, and the Rank Effect

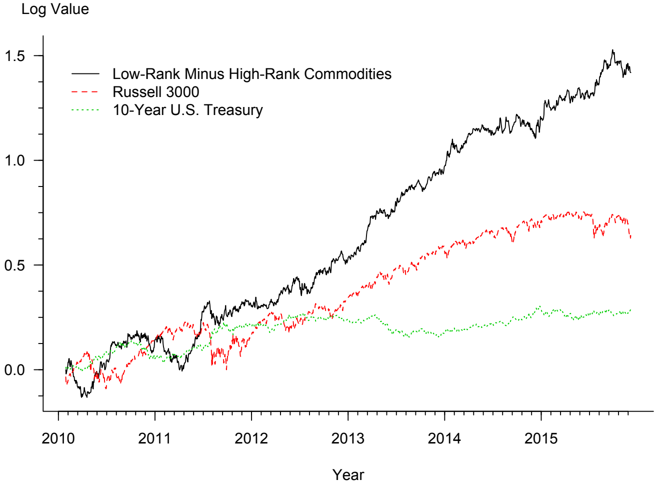

Figure: Log returns for rank effect, stocks, and bonds, 2010 - 2016.

A Structural Rank Effect?

- Rank effect doesn't appear to be driven by excess risk
  - Little apparent correlation between low-minus-high rank effect for commodities and U.S. equity returns or U.S. business cycle
  - Standard asset pricing theories predict correlation between low-minus-high rank effect and some discount factor (Lucas, 1978)
- Econometric theory predicts only a rank effect, but says nothing about the risk properties of that rank effect
  - Difficult to see how actions of investors can alter prices in a way that eliminates the rank effect
  - This points to a systematic relationship between rank and risk

Extensions and Applications

- Empirical methods for dynamic power law distributions
  - Methods can be applied to many different power law distributions
  - Nonparametric techniques are flexible and robust
- Other applications
  - Wealth and income: Fernholz (2016, 2017)
  - Firm size: Smaller firms generate faster employment growth
  - Historical commodity prices: 250 years of predictability?
  - World income distribution: Are we converging, and if so, to what?
  - City size: Similar to Gabaix (1999), but with more flexibility

Thank You