Heterogeneous Random Growth

RDS Research Workshop

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Homogeneous Random Growth

- Standard theory of random growth assumes homogeneous agents

$$\log x_i(t+1) - \log x_i(t) = \alpha + \sigma B_i(t),$$

  where $\sigma > 0$ and $B_i(t) \sim \mathbb{N}(0,1)$

- Many applications for standard theory of random growth
  - Firm size: Luttmer (QJE 2007)
  - City size: Gabaix (QJE 1999)
  - Income, wealth distributions: Gabaix, Lasry, Lyons, Moll (ECMA 2018)

Heterogeneous Random Growth

- Need to extend standard theory to match certain aspects of the data
  - Introduce permanent heterogeneity in the growth rates of agents

$$\log x_i(t+1) - \log x_i(t) = \alpha + \gamma_i + \sigma B_i(t)$$

- Two applications
  - Long-run wealth mobility: simultaneously match high short-run mobility and "low" long-run mobility (Benhabib, Bisin, and Fernholz, 2021)
  - City size: match many of the empirical findings of Davis and Weinstein (2002), also non-standard size distributions documented by Soo (2005)

Homogeneous Rank-Based Random Growth

For each unit i, size dynamics are given by

$$d \log x_i(t) = \alpha_{r_t(i)}\, dt + \sigma_{r_t(i)}\, dB_i(t),$$

where $r_t(i)$ is the rank of unit i at time t.

- Refer to size of units, can be wealth of households, size of cities, etc.
- Rank-based approach: $\log x_i(t+1) - \log x_i(t) = \alpha_k + \sigma_k B_i(t)$, with $B_i(t) \sim \mathbb{N}(0,1)$
- Easy to characterize stationary power law distribution in terms of parameters α_k and σ_k (Fernholz, 2017)

Proposition (Mobility and Homogeneity)

In the homogeneous rank-based model with N total units, for each unit i, asymptotic rank satisfies

$$\lim_{\tau \to \infty} \mathbb{E}[r_{t+\tau}(i)] = \frac{N+1}{2}.$$

- All units are ex-ante identical, so in the long run they will all approach the same rank — the median of the distribution
- No long-run persistence

Heterogeneous Rank-Based Random Growth

For each unit i, size dynamics are given by

$$d\log x_i(t) = \alpha_{r_t(i)}\,dt + \gamma_i\,dt + \sigma_{r_t(i)}\,dB_i(t),$$

where $r_t(i)$ is the rank of unit i at time t. For simplicity, assume $\gamma_i \in \{\gamma_\ell, \gamma_h\}$ with $\gamma_\ell < \gamma_h$ (two types: low-growth and high-growth).

This is the same setup as before, but with permanent heterogeneity:

$$\log x_i(t+1) - \log x_i(t) = \alpha_k + \gamma_i + \sigma_k B_i(t),$$

with $B_i(t) \sim \mathbb{N}(0,1)$.

Theorem (Mobility and Heterogeneity)

In the heterogeneous rank-based model, for all units i, j, asymptotic rank satisfies

$$\lim_{\tau\to\infty} \mathbb{E}[r_{t+\tau}(i)] < \lim_{\tau\to\infty} \mathbb{E}[r_{t+\tau}(j)] \qquad \textit{if and only if} \qquad r_t(i) < r_t(j).$$

- The higher the rank today, the higher the expected future rank
- High-growth units spend more time in higher ranks, on average, so the higher the rank today, the more likely a unit is to be a high-growth type (and hence the higher the expected future rank)
- Long-run persistence of ranks

Theorem (Distribution and Heterogeneity)

Suppose that there is only one high-growth unit with $\gamma_i = \gamma_h$. *Then, under certain conditions, the stationary size distribution satisfies*

$$\lim_{\gamma_h \to \infty} \mathbb{E}[\log x_{(1)}(t) - \log x_{(2)}(t)] = \infty,$$

and, for all $k > 1$,

$$\lim_{\gamma_h \to \infty} \mathbb{E}[\log x_{(k)}(t) - \log x_{(k+1)}(t)] = constant.$$

- As heterogeneity grows, on average the top-ranked unit grows relative to other units (top rank will also be highly persistent)
- Outside of top rank, distribution looks like standard power law

Short-Run and Long-Run Wealth Mobility

- Standard random growth models of wealth distribution do a poor job of matching long-run mobility (Benhabib, Bisin, and Fernholz 2021)
  - Wealth-rank coefficient after 585 years is 0.1: Barone & Mocetti (2021)
  - Both parent and grandparent wealth-rank have predictive power for child wealth-rank: Boserup, Kopczuk, & Kreiner (2014)

- Can a model with permanent heterogeneity simultaneously match wealth distribution, short-run mobility, and long-run mobility?

Homogeneous Rank-Based Model

Build a microfounded, incomplete markets model based on the models of Benhabib et al. (2011, 2019), with intergenerational wealth dynamics

$$x_i(t+1) = \lambda x_i(t) + \beta, \tag{1}$$

where λ and β are endogenous and depend on structural parameters (returns, income, preferences, policy) of model.

Following Fernholz (2017), estimate a rank-based model based on (1),

$$d \log x_i(t) = \alpha_k \, dt + \sigma_k \, dB_i(t),$$

where parameters α_k and σ_k depend on λ and β.

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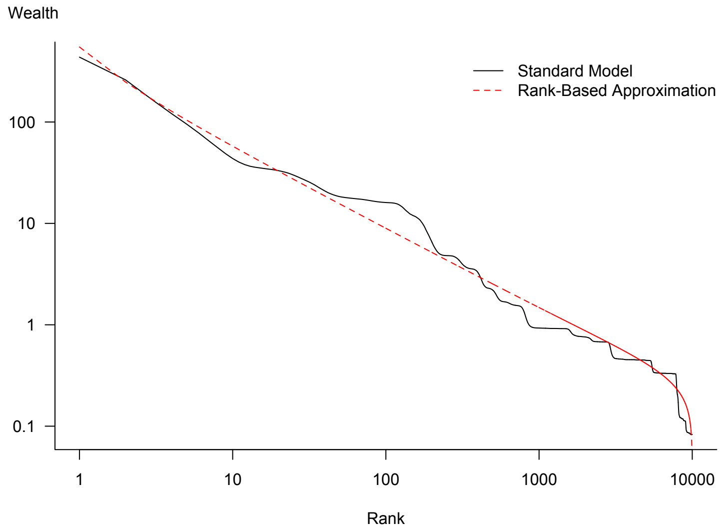
Rank-Based Model



Heterogeneous Rank-Based Model

Also consider a rank-based model with permanent heterogeneity,

$$d \log x_i(t) = \alpha_k \, dt + \gamma_i \, dt + \sigma_k \, dB_i(t),$$

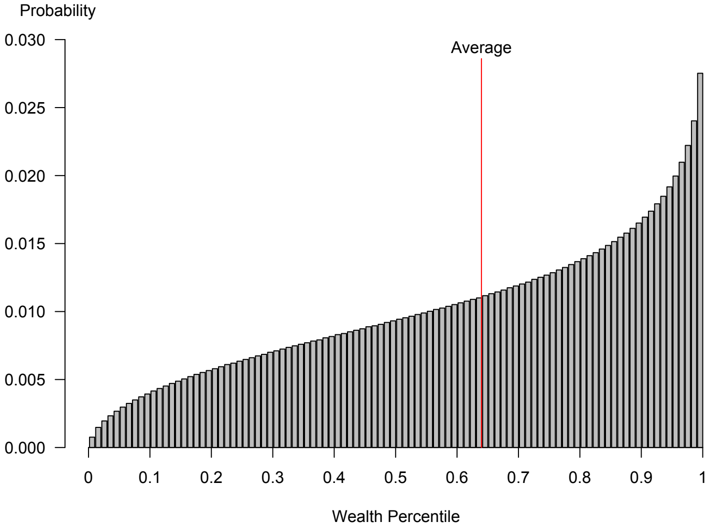
where $\gamma_i \in \{\gamma_\ell, \gamma_h\}$ with $\gamma_\ell < \gamma_h$ (two types of households: low-growth and high-growth).

This permanently heterogeneous model is constructed to generate (approximately) the same wealth distribution as the homogeneous model.

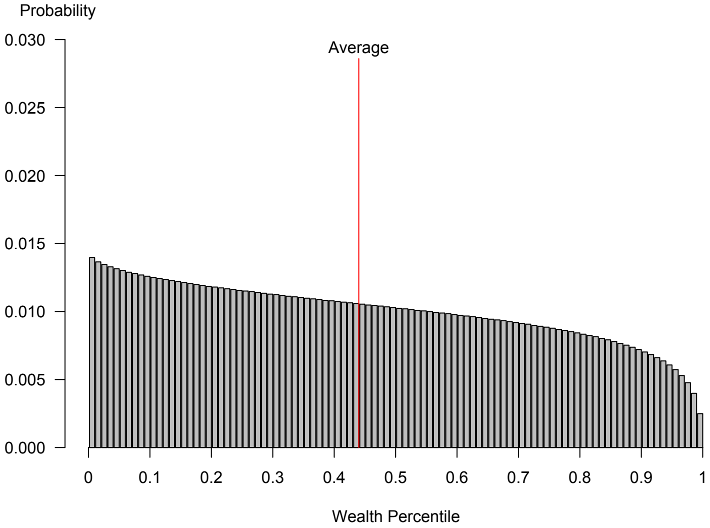
Homogeneous vs. Heterogeneous Models

	Data	Approximated Rank-Based Model	Perman. Heterog. Rank-Based Model
Wealth Distribution			
Top 1%	33.6%	31.9%	34.0%
Top 1-5%	26.7%	17.1%	16.6%
Top 5-10%	11.1%	9.5%	9.2%
Top 10-20%	12.0%	11.2%	10.8%
Top 20-40%	11.2%	13.1%	12.7 %
Top 40-60%	4.5%	8.3%	8.1%
Bottom 40%	-0.1%	8.9%	8.5%
Wealth-Rank Correlations			
Parent-Child Rank Coeff.	0.191	0.229	0.255
Grandparent-Child Rank Coeff.	0.116	0.018	0.077
Long-Run Persistence Coeff.	0.105	0.000	0.100

High-Growth Percentiles



Low-Growth Percentiles



Auto-Correlated Returns

Recall the original incomplete markets model with intergenerational wealth dynamics

$$x_i(t+1) = \lambda x_i(t) + \beta.$$

If intergenerational wealth returns (and income) follow an AR-1 process, then this creates intergenerational persistence in the parameters λ and β as well. Can this model match the data?

Auto-Correlated vs. Permanently Heterogeneous Models

	Data	Auto-Correlated Returns Model ($\theta = 0.95$)	Perman. Heterog. Rank-Based Model
Wealth Distribution			
Top 1%	33.6%	31.5%	34.0%
Top 1-5%	26.7%	20.6%	16.6%
Top 5-10%	11.1%	12.3%	9.2%
Top 10-20%	12.0%	13.5%	10.8%
Top 20-40%	11.2%	12.8%	12.7%
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Parent-Child Rank Coeff.	0.191	0.407	0.255
Grandparent-Child Rank Coeff.	0.116	0.044	0.077
Long-Run Persistence Coeff.	0.105	0.041	0.100

Auto-Correlated Returns

- Auto-correlated returns model does not match long-run mobility data as well as permanently heterogeneous model
  - Both grandparent-child rank and long-run rank persistence coefficients are too low
- Auto-correlated returns model also generates too much intergenerational return rank persistence
  - Auto-correlated returns model predicts 0.95, while Fagereng, Guiso, Malacrino, & Pistaferri (2020) find 0.16 in Norwegian data
  - Permanently heterogeneous model predicts 0.08

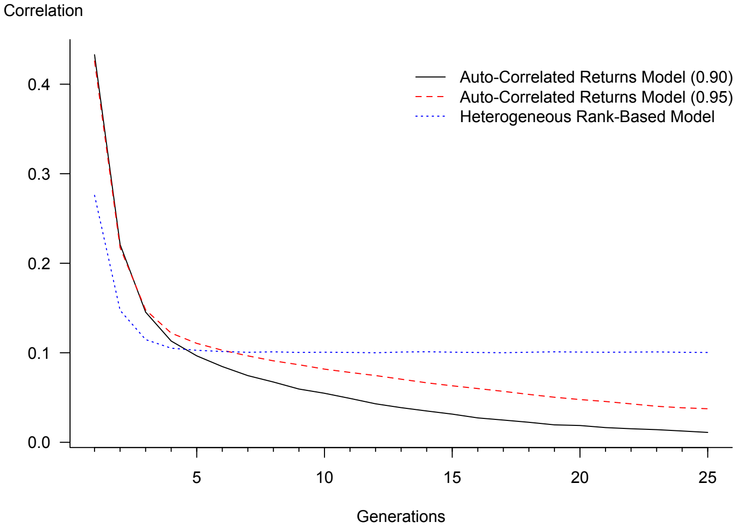
Intuition

- How does the permanently heterogeneous model match the grandparent-child rank coefficient?
  - The higher the grandparent wealth-rank, the more likely a child is to be a high-growth household and hence the higher its expected wealth-rank
  - True even controlling for parent wealth-rank (non-Markovian model)
- How does the permanently heterogeneous model match long-run persistence of rank?
  - Mobility and heterogeneity theorem:

$$\lim_{\tau \to \infty} \mathbb{E}[r_{t+\tau}(i)] < \lim_{\tau \to \infty} \mathbb{E}[r_{t+\tau}(j)] \quad \text{if and only if} \quad r_t(i) < r_t(j).$$

  - Non-ergodic model

Long-Run Persistence



Questions

$$d \log x_i(t) = \alpha_k \, dt + \gamma_i \, dt + \sigma_k \, dB_i(t)$$

- What do the γ_i terms capture?
  - Persistent cultural and institutional factors
  - Low parent-child return-rank correlation points to persistent institutional factors for sustaining long-run wealth persistence
- A more careful quantitative calibration?
  - Build richer model and calibrate by targeting specific moments
  - Difficult to find data on distribution, wealth-rank and return-rank correlations, and mobility all from one country

Different Theories

Davis and Weinstein (2002) consider three competing theories

1. (Homogeneous) Random growth: cities grow randomly, and this leads to a Pareto distribution and Zipf's law

2. Increasing returns: big cities have advantages because of knowledge spillovers, labor-market pooling

3. Locational fundamentals: fundamental economic characteristics are random, and determine size

Post-WW II Japan suggests growth is not fully random, since same cities as before grow to be largest

Random Growth and Locational Fundamentals

- Combine random growth and locational fundamentals theories:

$$d \log x_i(t) = \alpha_k \, dt + \gamma_i \, dt + \sigma_k \, dB_i(t)$$

  - Locational fundamentals imply permanent heterogeneity via γ_i terms

- Mobility and heterogeneity theorem matches post-WW II Japanese experience:

$$\lim_{\tau\to\infty} \mathbb{E}[r_{t+\tau}(i)] < \lim_{\tau\to\infty} \mathbb{E}[r_{t+\tau}(j)] \qquad \text{if and only if} \qquad r_t(i) < r_t(j).$$

  - City ranks are persistent, so larger cities pre-WW II are expected to grow larger again despite their destruction during WW II

Deviations from Power Laws

- Many deviations from Zipf's law and power laws for city size distributions around the world (Gabaix, 1999; Soo, 2005)
  - Biggest city or cities often "too large": France, UK, South Korea, Mexico, Argentina, Russia

- Distribution and heterogeneity theorem matches the deviations. If $d \log x_i(t) = \alpha_k\, dt + \gamma_i\, dt + \sigma_k\, dB_i(t)$, with $\gamma_i \in \{\gamma_\ell, \gamma_h\}$ and $\gamma_\ell < \gamma_h$, then

$$\lim_{\gamma_h \to \infty} \mathbb{E}[\log x_{(1)}(t) - \log x_{(2)}(t)] = \infty,$$

while the rest of the distribution looks like a standard power law
  - Top-ranked city is different from the rest, persists at the top

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Questions

$$d \log x_i(t) = \alpha_k \, dt + \gamma_i \, dt + \sigma_k \, dB_i(t)$$

- How important is permanent heterogeneity?
  - What is the magnitude of γ_i parameters?
  - How many different types are needed to match the data?
- Are there other patterns in the data that the combined random growth and locational fundamentals model can or cannot explain?
  - Changes in city size distributions over time
- What about time-varying growth rates and volatilities, or location-specific variances (i.e. σ_i)?

The End

Thank You