A Random Growth Approach to Asset Returns

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Random Growth in Economics

Random growth theory posits that entities $X_1, \dots, X_n$ grow according to

$$\log X_i(t+1) - \log X_i(t) = g_i(t) + \sigma_i(t)\, B_i(t),$$

where $\sigma_i(t) > 0$ and $B_i(t) \sim \mathbb{N}(0,1)$. In continuous time, this becomes

$$d \log X_i(t) = g_i(t)\, dt + \sigma_i(t)\, dB_i(t).$$

- Many applications of random growth in economics
  - Firm size: Luttmer (QJE 2007)
  - City size: Gabaix (QJE 1999)
  - Income, wealth distributions: Gabaix, Lasry, Lyons, Moll (ECMA 2018)

Random Growth in Finance?

- Let $X_1, \dots, X_n$ be the value of assets in a financial market

$$d \log X_i(t) = g_i(t)\,dt + \sigma_i(t)\,dB_i(t)$$

- Asset values $X_1, \dots, X_n$ change over time, and these changes generate capital gains and hence contribute to returns
  - Are dynamics of X_i endogenous or exogenous?

- Parameters g_i and σ_i affect both asset returns and the distribution of asset values

$$\frac{dX_i(t)}{X_i(t)} = \left(g_i(t) + \frac{\sigma_i^2(t)}{2}\right)\,dt + \sigma_i(t)\,dB_i(t)$$

Rank-Based Random Growth

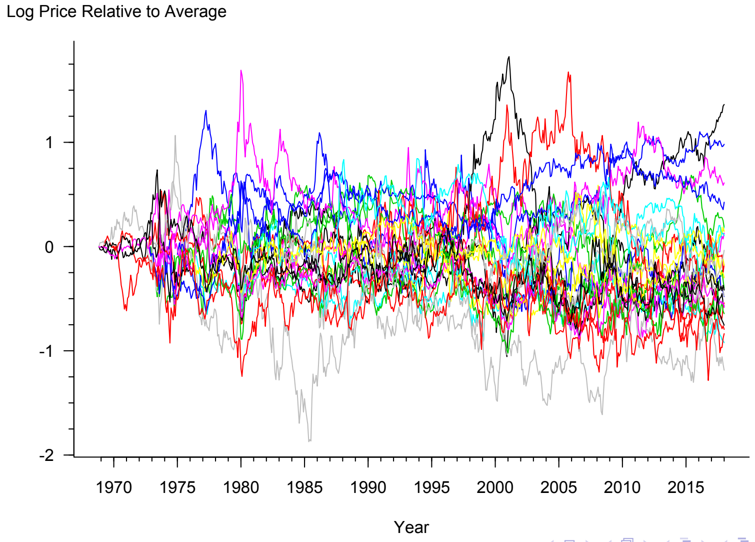
$$d \log X_i(t) = g_i(t)\, dt + \sigma_i(t)\, dB_i(t)$$

- Parameters g_i and σ_i depend on many economic and financial factors, and are changing over time
  - Very difficult to accurately estimate parameters
- Consider simple random growth model in which parameters g_i and σ_i depend only on asset value rank
  - Rank-based parameters g_k and σ_k shape distribution (Fernholz, 2017)
  - If distribution is stable, rank-based parameters should also be stable

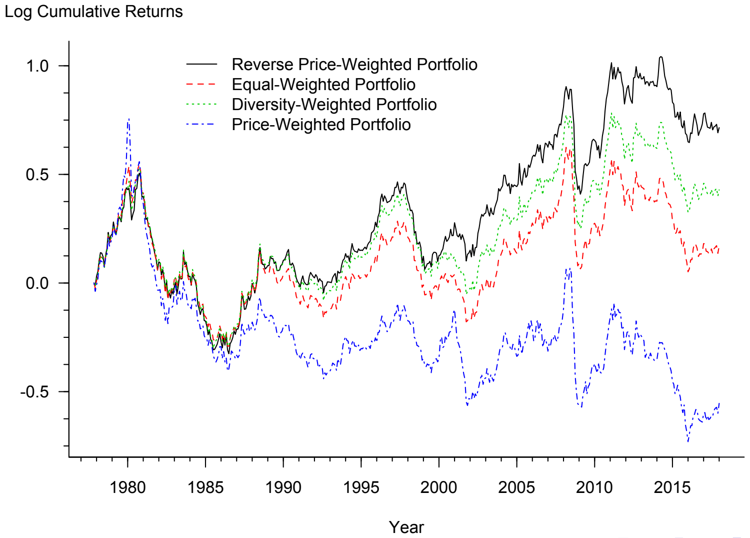
Applications

- Normalized commodity futures prices
  - Distribution of commodity values is stable over time, which implies higher growth rates at lower ranks
  - Leads to predictable excess returns (Fernholz & Fernholz, 2022)
- Market capitalizations of U.S. stocks
  - Distribution of stock market capitalizations is stable over time and well-described by rank-based parameters (Fernholz & Karatzas, 2009)
  - Parameters both shape distribution and lead to well-known size effect
- Extensions of simple rank-based random growth model
  - Ichiba et al. (2011), Benhabib, Bisin, & Fernholz (2022)

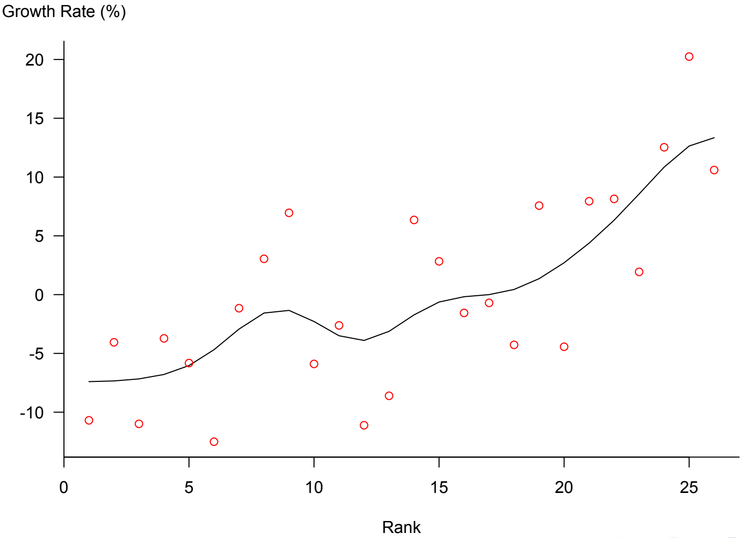
Normalized Commodity Futures Prices



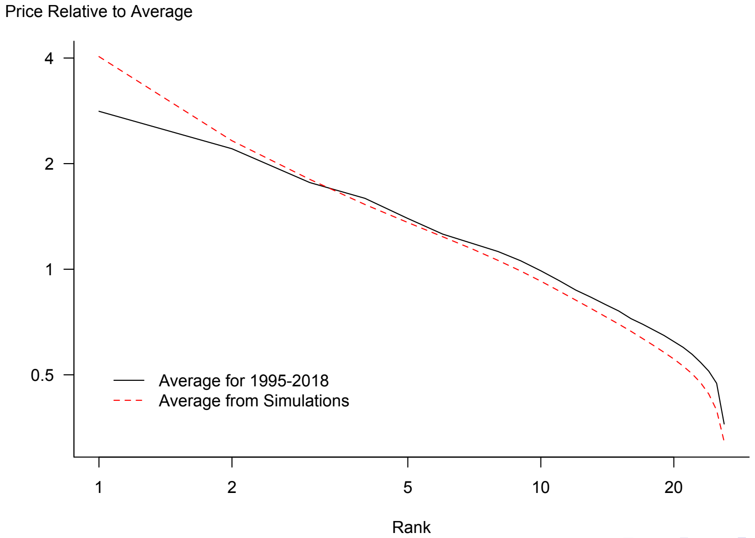
Returns for Portfolios of Commodity Futures



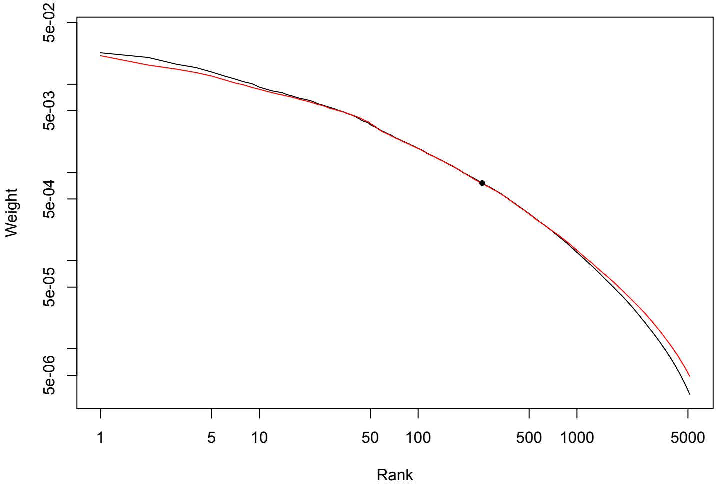
Rank-Based Growth Rates



Predicted vs. Actual Distribution of Commodity Values



Predicted vs. Actual Distribution of Market Capitalizations



Actual (black), predicted (red)

Rank-Based Random Growth

Consider a financial market consisting of n assets with values $X_1, \ldots, X_n$ that grow according to

$$d \log X_i(t) = g_{r_t(i)}(t)\, dt + \sigma_{r_t(i)}(t)\, dB_i(t),$$

where $r_t(i)$ is the value-rank of asset i at time t, $\sigma_1^2, \ldots, \sigma_n^2$ are positive constants, and $g_1, \ldots, g_n$ are constants satisfying $g_1 + \cdots + g_n = 0$.

- Parameters g_k measure relative growth rates at different ranks (normalization $g_1 + \cdots + g_n = 0$ is without loss of generality)
- Parameters σ_k measure variance at different ranks

Stationary Distribution of Relative Values

$$d \log X_i(t) = g_{r_t(i)}(t)\, dt + \sigma_{r_t(i)}(t)\, dB_i(t) \tag{1}$$

Suppose that the model (**??**) satisfies $g_1 + \cdots + g_k < 0$ for all $k = 1, \ldots, n-1$ and $\sigma_{k+1}^2 - \sigma_k^2 = \sigma_k^2 - \sigma_{k-1}^2$, for all $k = 2, \ldots, n-1$.

Then the ranked relative values are stationary and satisfy

$$\mathbb{E}[\log X_{(k)}(t) - \log X_{(k+1)}(t)] = \frac{\sigma_k^2 + \sigma_{k+1}^2}{-4(g_1 + \cdots + g_k)},$$

for all $k = 1, \ldots, n-1$ (Ichiba et al., 2011). Note that $X_{(1)}, \ldots, X_{(n)}$ are ranked by value, so $X_{(1)} \geq X_{(2)} \geq \cdots \geq X_{(n)}$.

Stationary Asset Values and Efficient Markets

$$\frac{dX_i(t)}{X_i(t)} = \left(g_{r_t(i)}(t) + \frac{\sigma^2_{r_t(i)}}{2} \right) dt + \sigma_{r_t(i)}(t)\, dB_i(t),$$

A stationary, or at least non-degenerate, relative value distribution seems reasonable. How can such a market be efficient?

- Entry/Exit
  - If assets enter and exit with sufficient frequency, then do not need higher growth rates at lower ranks (Fernholz & Koch, 2021)
  - Important consideration for equities, less so for commodity futures
- Dividends
  - Higher-ranked assets can pay more dividends

Stationary Asset Values and Efficient Markets

$$\frac{dX_i(t)}{X_i(t)} = \left(g_{r_t(i)}(t) + \frac{\sigma^2_{r_t(i)}}{2} \right) dt + \sigma_{r_t(i)}(t)\, dB_i(t),$$

A stationary, or at least non-degenerate, relative value distribution seems reasonable. How can such a market be efficient?

- Risk
  - Lower-ranked assets can be riskier, so higher returns on these assets compensate for their greater risk
  - Asset rank, with rank based on asset value, as a risk factor (Fama & French, 1992; Asness et al., 2013)

"Value" of Commodity Futures

- Monthly futures prices for 26 commodities from 1969-2018
  - Focus on two-month futures contracts, which are among most liquid
  - Commodity futures do not pay dividends and rarely "exit"
- Value of two-month commodity futures is normalized price
  - Set all futures values equal to each other on first month, with all subsequent changes in (log) value equal to changes in (log) price
  - Commodities that enter later have (log) value set equal to average
  - Similar to Asness, Moskowitz, & Pedersen (JF 2013)

Implied Commodity Futures Prices

Not all commodities have two-month futures contracts in all months, so we define implied two-month futures prices that exist in all months.

The *implied two-month futures price* at time t for commodity i is

$$\log X_i(t) = (2-\nu)\kappa_i(t) + \log F_i(t, t+\nu),$$

where $F_i(t, t+\nu)$ is the futures price at time t with expiration $t+\nu$ (closest possible expiration to $t+2$), and

$$\kappa_i(t) = \frac{\log F_i(t, t+\nu_2) - \log F_i(t, t+\nu_1)}{\nu_2 - \nu_1},$$

with $t+\nu_1$ and $t+\nu_2$ expiration dates closest to $t+2$.

Implied Commodity Futures Prices

The *implied two-month futures price* at time t for commodity i is

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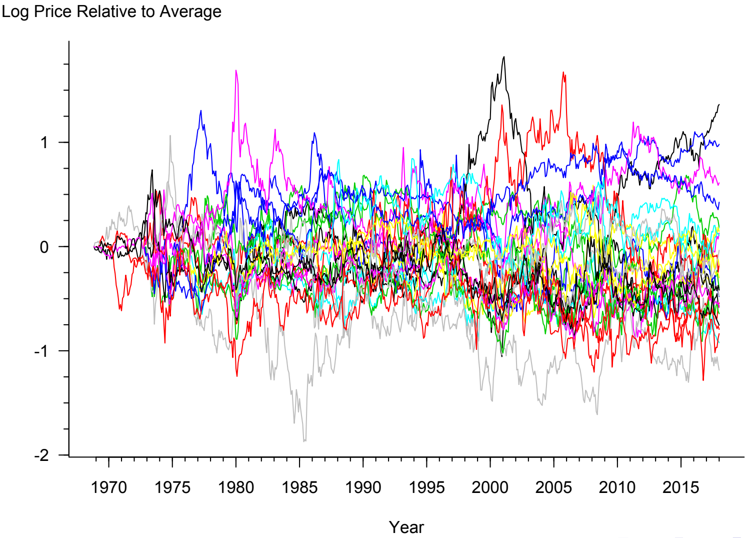
with $t + \nu_1$ and $t + \nu_2$ expiration dates closest to $t + 2$.

Implied two-month futures price adjusts and interpolates using existing futures contracts. Note that if two-month futures contract exists, then implied price equals actual price: $X_i(t) = F_i(t, t + 2)$.

Implied Two-Month Commodity Futures Prices



Portfolios of Commodity Futures

- Four portfolios, each rebalanced monthly
  - Value-weighted: $w_i(t) = \frac{X_i(t)}{X_1(t)+\cdots+X_n(t)}$
  - Equal-weighted: $w_i(t) = \frac{1}{n}$
  - Diversity-weighted: $w_i(t) = \frac{X_i^p(t)}{X_1^p(t)+\cdots+X_n^p(t)}$, with $p = -0.5$
  - Reverse value-weighted: $w_i(t) = \frac{X_{(n+1-r_t(i))}(t)}{X_1(t)+\cdots+X_n(t)}$ (need to introduce $r_t(i)$)
- Value-weighted portfolio places most weight on high ranks, while reverse-weighted portfolio places most weight on low ranks
  - Vervuurt & Karatzas (2015) examine diversity-weighted prt. with $p < 0$
  - Fernholz & Fernholz (2022) examine reverse-weighted portfolio

Portfolios of Commodity Futures

- Portfolios all hold two-month futures contracts if possible
  - If not, hold the contract with the next expiration horizon greater than two months
  - Commodity values are normalized prices, so wait five years after price data start date before including a commodity in portfolios
  - All portfolios are rebalanced monthly, so they should all have similar transaction costs
  - Similar to Asness, Moskowitz, & Pedersen (JF 2013)

Carry

The change in the implied two-month futures price, $\Delta \log X_i(t)$, is not necessarily equal to the return from holding the underlying commodity futures contract, $\Delta \log F_i(t,\tau)$, where $\tau \geq t + 2$.

We refer to the difference between these two quantities as the *carry*:

$$C_i(t) = \Delta \log F_i(t,\tau) - \Delta \log X_i(t).$$

The carry measures the gap between the returns from holding commodity futures contracts and changes in the implied futures prices.

Carry and Market Efficiency

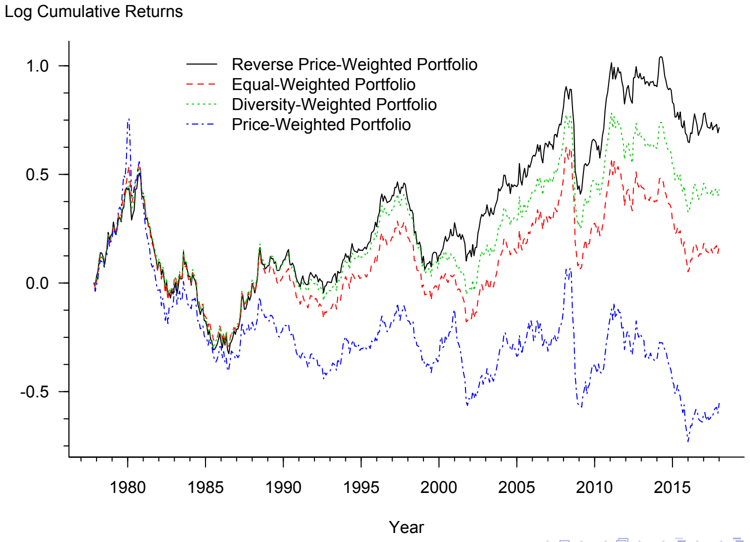
$$\Delta \log F_i(t, \tau) = \Delta \log X_i(t) + C_i(t)$$

- Rank-based theory suggests low-ranked commodity values, X_i, grow faster than high-ranked values
  - Necessary for a stationary value distribution in the absence of entry/exit
  - Points to higher returns at low ranks in the absence of dividends
- How can this market be efficient?
  - Need more negative carry at lower ranks
  - Assuming risk properties at high and low ranks are similar

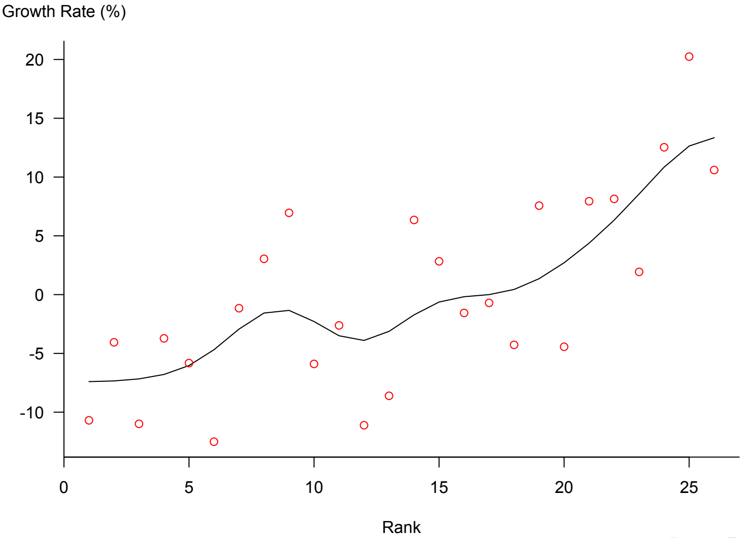
Returns for Portfolios of Commodity Futures

	Price-Weighted	Equal-Weighted	Diversity-Weighted	Reverse-Weighted
Average	-1.43%	0.43%	1.09%	1.83%
Standard Deviation	15.38%	13.79%	13.68%	13.85%
CAPM Beta	0.11	0.07	0.06	0.05
Sharpe Ratio		0.40	0.41	0.47

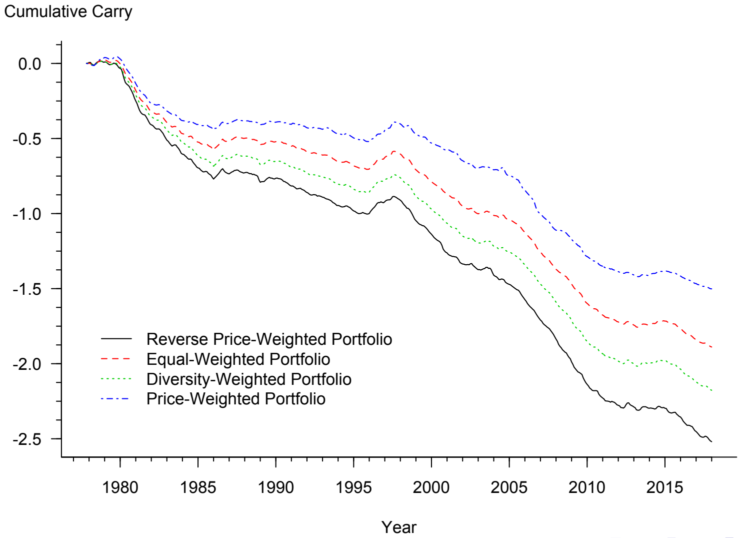
Returns for Portfolios of Commodity Futures



Rank-Based Growth Rates



Carry


Cumulative Carry
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
1980
1985
1990
1995
2000
2005
2010
2015
Year
Reverse Price-Weighted Portfolio
Equal-Weighted Portfolio
Diversity-Weighted Portfolio
Price-Weighted Portfolio

Interpretation

- Asness, Moskowitz, & Pedersen (JF 2013) find a similar result
  - Rank commodity futures based on current price relative to average price 4.5-5.5 years ago
  - High "value" commodities outperform low "value" commodities
  - Posit a general value factor affecting many different asset markets
- Rank-based random growth model has complementary interpretation
  - Low-ranked (high-value) commodities must grow faster for stationarity
  - Puzzle is that differential carry does not cancel out the higher growth rate of low-ranked commodities
  - Any risk factor should explain differential carry that is too small

The Distribution of Relative Commodity Values

For a rank-based random growth model of the form

$$d \log X_i(t) = g_{r_t(i)}(t)\, dt + \sigma_{r_t(i)}(t)\, dB_i(t),$$

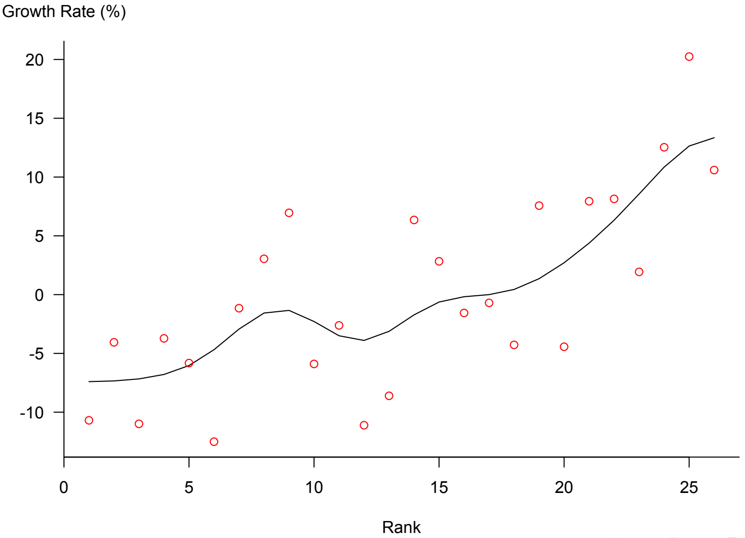
the stationary ranked relative values satisfy

$$\mathbb{E}[\log X_{(k)}(t) - \log X_{(k+1)}(t)] = \frac{\sigma_k^2 + \sigma_{k+1}^2}{-4(g_1 + \cdots + g_k)},$$

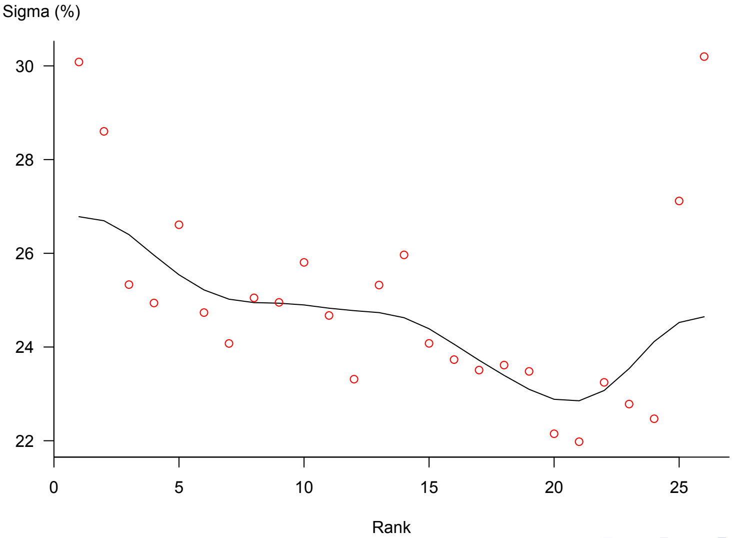
for all $k = 1, \ldots, n-1$.

Follow the procedure described by Fernholz (2017) to estimate the rank-based parameters g_k and σ_k for implied two-month futures prices from 1995-2018.

Rank-Based Growth Rates



Rank-Based Variances



Simulations of Rank-Based Model

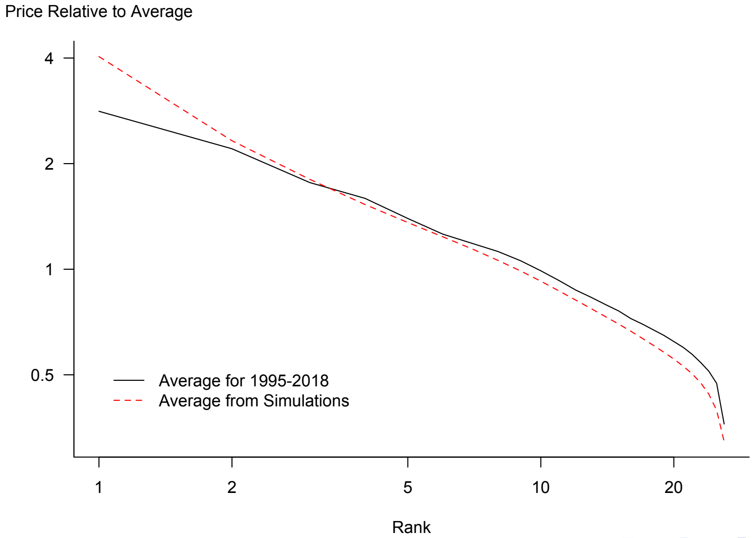
If $g_1 + \cdots + g_k < 0$ for all $k = 1, \ldots, n-1$ and $\sigma_{k+1}^2 - \sigma_k^2 = \sigma_k^2 - \sigma_{k-1}^2$, for all $k = 2, \ldots, n-1$, then the stationary ranked relative values are stationary and satisfy

$$\mathbb{E}[\log X_{(k)}(t) - \log X_{(k+1)}(t)] = \frac{\sigma_k^2 + \sigma_{k+1}^2}{-4(g_1 + \cdots + g_k)}, \tag{2}$$

for all $k = 1, \ldots, n-1$.

However, the estimated parameters σ_k do not clearly satisfy the linearity condition for (**??**). The solution is to simulate a rank-based model with the esitmated parameters g_k and σ_k.

Simulated vs. Actual Distribution of Commodity Values



Stock Returns and the Distribution of Market Caps

$$d\log X_i(t) = g_{r_t(i)}(t)\,dt + \sigma_{r_t(i)}(t)\,dB_i(t),$$

- One insight of rank-based random growth model is link between distribution of relative asset values and asset returns
  - Rank-based parameters g_k and σ_k shape the relative value distribution and also affect returns
- For stocks, this insight implies a link between distribution of stock market capitalizations and stock returns
  - Capital gains lead to changes in market capitalization of stocks
  - Dividends also impact stocks returns, but capital gains more important

Stock Returns and the Distribution of Market Caps

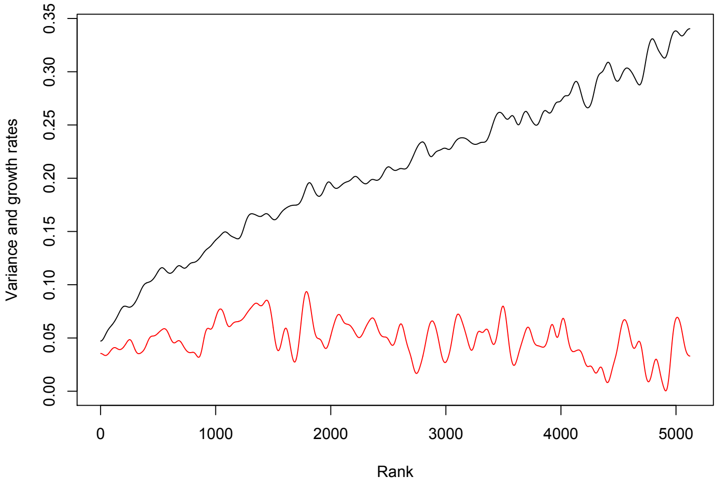
Let $X_1, \ldots, X_n$ denote the market capitalizations (values) of the n stocks in the market. If certain regularity conditions are satisfied, then the stationary distribution of market caps satisfies

$$\mathbb{E}[\log X_{(k)}(t) - \log X_{(k+1)}(t)] = \frac{\sigma_k^2 + \sigma_{k+1}^2}{-4(g_1 + \cdots + g_k)},$$

for all $k = 1, \ldots, n-1$.

Follow the procedure described by Fernholz & Koch (2021) to estimate the rank-based parameters g_k and σ_k for U.S. stocks from 1990-1999.

Rank-Based Parameters for Market Capitalizations



σ_k^2 (black), $-g_k$ (red)

Stock Returns and the Distribution of Market Caps

Let $X_1, \dots, X_n$ denote the market capitalizations (values) of the n stocks in the market. If the rank-based growth rates satisfy $g_1 = \cdots = g_{n-1} = g$, then the stationary distribution of market caps satisfies

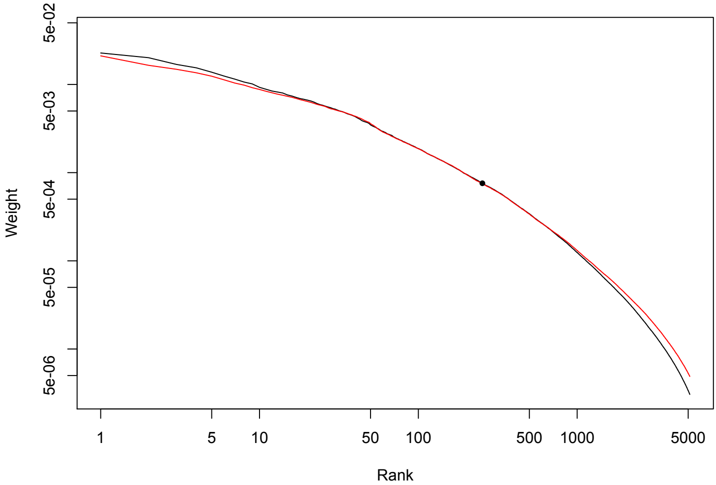
$$\frac{\mathbb{E}[\log X_{(k)}(t) - \log X_{(k+1)}(t)]}{\log(k) - \log(k+1)} \approx -\frac{k(\sigma_k^2 + \sigma_{k+1}^2)}{4kg} = -\frac{\sigma_k^2 + \sigma_{k+1}^2}{4g},$$

for all $k = 1, \dots, n-1$.

In a log-log plot of size vs. rank, stock market capitalization distribution curve will be concave if σ_k^2 is linearly increasing in k.

In this market, entry/exit of stocks ensures stationarity since growth rates are not higher at lower ranks (unlike closed commodity futures market).

Predicted vs. Actual Distribution of Market Capitalizations



Actual (black), predicted (red)

The Size-Effect Revisited

$$\frac{dX_i(t)}{X_i(t)} = \left(g_{r_t(i)}(t) + \frac{\sigma^2_{r_t(i)}}{2} \right) dt + \sigma_{r_t(i)}(t)\, dB_i(t),$$

- What do constant rank-based growth rates g_k and linearly increasing rank-based variances σ^2_k imply for stock returns?
  - Higher σ_k at low ranks implies higher returns at low ranks, all else equal
- Well-known size effect follows from shape of market cap distribution
  - Banz (1981), Fernholz & Karatzas (2009), Banner et al. (2019)

Interpretation

- Size effect for equities is a well-known anomaly
  - Many potential explanations: risk, liquidity, mismeasurement
  - Banz (1981), Fama & French (1992), Vayanos (2003), Van Dijk (2011)
  - Size (SMB) is an important and common factor in asset pricing models
- Rank-based random growth model offers a novel interpretation
  - Rank-based parameters g_k and σ_k shape the market cap distribution
  - These parameters also affect returns for different size-ranked stocks
  - Size risk factor is related to the shape of the size distribution
  - "Value" anomaly for commodities and size effect for stocks are related

A Model with Persistent Heterogeneity

The standard rank-based random growth model

$$d \log X_i(t) = g_{r_t(i)}(t)\, dt + \sigma_{r_t(i)}(t)\, dB_i(t), \tag{3}$$

seems to decently describe relative commodity price and stock market cap distributions. However, this model is overly simplistic in some ways.

The extension of the standard rank-based random growth model

$$d \log X_i(t) = \left(g_{r_t(i)}(t) + \gamma_i\right) dt + \sigma_{r_t(i)}(t)\, dB_i(t),$$

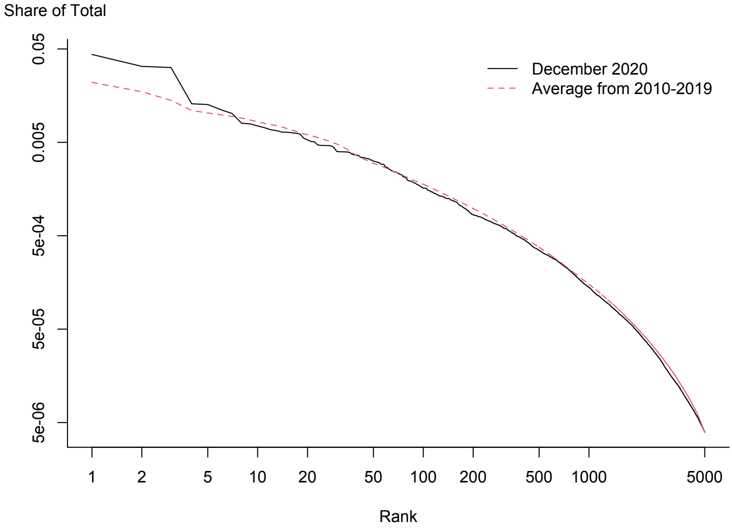
where $\sigma_1^2, \ldots, \sigma_n^2$ are positive constants and $g_1, \ldots, g_n, \gamma_1, \ldots, \gamma_n$ are constants satisfying two regularity conditions (Ichiba et al., 2011), captures richer dynamics than the standard model (**??**).

A Model with Persistent Heterogeneity

$$d \log X_i(t) = \left(g_{r_t(i)}(t) + \gamma_i\right) dt + \sigma_{r_t(i)}(t)\, dB_i(t)$$

- Parameters γ_i allow for persistent heterogeneity across assets/entities
  - Assets with high γ_i spend more time at high ranks
  - If assets with high γ_i grow more slowly than aggregate when in top ranks, then model is stationary (Ichiba et al., 2011)
  - Non-ergodic model
  - Accurately estimating all parameters from data can be quite challenging

U.S. Market Caps Distribution Pre-2020 vs. End-2020



Wealth Distribution and Long-Run Mobility

- Surprising findings for long-run mobility that are impossible to match using standard random growth models of wealth distribution
  - Wealth-rank coefficient after 585 years is 0.1: Barone & Mocetti (2021)
  - Both parent and grandparent wealth-rank have predictive power for child wealth-rank: Boserup, Kopczuk, & Kreiner (2014)
- Rank-based model of intergenerational wealth dynamics with persistent heterogeneity can match all of these observations
  - Benhabib, Bisin, & Fernholz (2022)

City Size Distributions

- According to Soo (2005), some city size distributions are neither Zipfian nor quasi-Zipfian
  - France, Argentina, Russia, Mexico, New York State, etc.
  - Largest cities are fundamentally, persistently different from the rest
- Davis & Weinstein (2002) show that after the destruction of WWII, the cities that grew to be largest were the same as those from before
  - Japanese city growth depends on size and locational fundamentals
- Both of these observations can be captured via a rank-based model with persistent heterogeneity

The End

Thank You