Instability and Concentration in the Distribution of Wealth

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Income and Wealth Distributions

- Significant concentration of income and wealth observed worldwide

- Richest 1% holds 33% of wealth, earns 25% of income in U.S.

- Gini coefficients of 0.55 for income and 0.80 for wealth in U.S.
  - Díaz-Giménez, Quadrini, Ríos-Rull, and Rodríguez (2002)
  - Davies, Sandström, Shorrocks, and Wolff (2011)

- Many different explanations have been proposed
Idiosyncratic Investment Risk

- Uninsurable idiosyncratic investment risk another possible explanation
  - Angeletos and Calvet (2006), Benhabib, Bisin, and Zhu (2011)
  - Generates realistic stationary Pareto distribution of wealth

- Strong empirical motivation for uninsurable investment risk
  - Together, these make up more than 50% of total U.S. household wealth: Bertaut and Starr-McCluer (2002) and Wolff (2012)
Key Elements of the Model

1. Heterogeneous households that face idiosyncratic investment risk
   - Can invest in individual-specific asset subject to uninsurable risk

2. No redistributive mechanisms
   - Broad interpretation: any factor that affects wealthy households and poor households differently
     - Government tax or fiscal policies
     - Limited intergenerational transfers

3. Forward-looking households that behave optimally
   - Choose how much to consume and how to invest savings
   - Efficient and rational model outcome
Instability and Concentration

- Equilibrium distribution of wealth is not stationary
  - Right-skewness increases over time
  - Wealth eventually concentrates entirely at the top
  - Instability characterized using recent results from mathematical finance

- Luck alone generates diverging levels of wealth
  - Households have same abilities, opportunities, and preferences

- Unlike previous literature, no redistributive mechanisms in this setup
  - Explicit, implicit redistributive mechanisms play crucial stabilizing role
Equilibrium Wealth Dynamics

Figure: The shares of total wealth held by the wealthiest 1% (solid black line), the wealthiest 1-5% (dotted red line), and the wealthiest 5-10% (dashed green line).
Outline

1 Introduction
   • Motivation
   • Preview of Results

2 The Model
   • Setup
   • Equilibrium Dynamics
   • Inequality and Idiosyncratic Investment Risk

3 Conclusion
Heterogeneous Households

- Economy is populated by \( N \) infinitely-lived households
  - At each \( t \in [0, \infty) \), each household solves savings-consumption problem
  - Each household receives labor income equal to \( \lambda \) throughout its life

- Households have two investment options:
  - Risk-free asset that pays a return of \( r \)
  - Individual-specific asset subject to idiosyncratic risk

- For all \( i = 1, \ldots, N \), price of individual-specific risky asset given by

\[
dP_i(t) = \alpha P_i(t) \, dt + \sigma P_i(t) \, dB_i(t)
\]

  - Uninsurable risk: Brownian motions \( B_i \) independent of each other
Heterogeneous Households

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  - At each $t \in [0, \infty)$, each household solves savings-consumption problem
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  \[ dP_i(t) = \alpha P_i(t) \, dt + \sigma P_i(t) \, dB_i(t) \]
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Household Maximization Problems

Each household $i$ has CRRA utility from consumption:

$$J(w, t) = \max_{c_i(t), \phi_i(t)} \mathbb{E}_t \left[ \int_t^\infty \frac{c_i^{1-\gamma}(s)}{1-\gamma} e^{-\rho s} \, ds \right]$$

s.t. $dw_i(s) = [rw_i(s) + (\alpha - r)\phi_i(s)w_i(s) - c_i(s) + \lambda] \, ds$

$$+ \sigma \phi_i(s)w_i(s) \, dB_i(s)$$

$c_i(t)$: Consumption

$\phi_i(t)$: Fraction of wealth $w_i(t)$ invested in risky asset

$\gamma \geq 1$: Coefficient of relative risk aversion

$\rho > 0$: Discount rate
Consumption and Investment

Proposition

For all households $i = 1, \ldots, N$, the policy functions $c_i(t)$ and $\phi_i(t)$ are given by

$$c_i(t) = \left( \frac{\rho - (1 - \gamma)r}{\gamma} - \frac{(1 - \gamma)(\alpha - r)^2}{2\gamma^2\sigma^2} \right) \left( w_i(t) + \frac{\lambda}{r} \right),$$

$$\phi_i(t) = \frac{(\alpha - r) \left( w_i(t) + \frac{\lambda}{r} \right)}{w_i(t)\gamma\sigma^2}.$$

- Household maximization problems similar to Merton (1969)
  - Standard intuition behind optimal $c_i(t)$ and $\phi_i(t)$
  - $\frac{\lambda}{r}$ equal to households’ discounted future labor income
Household Wealth Dynamics

Let $x_i(t)$ be total wealth of household $i = 1, \ldots, N$ at time $t$, so that

$$x_i(t) = w_i(t) + \frac{\lambda}{r}.$$ 

Given $c_i(t)$ and $\phi_i(t)$, then

$$dx_i(t) = \left( \frac{r - \rho}{\gamma} + \frac{(1 + \gamma)(\alpha - r)^2}{2\gamma^2\sigma^2} \right) x_i(t) \, dt + \left( \frac{\alpha - r}{\gamma\sigma} \right) x_i(t) \, dB_i(t).$$

Substitution then yields

$$dx_i(t) = \Lambda x_i(t) \, dt + \Gamma x_i(t) \, dB_i(t),$$

so each household’s wealth follows geometric Brownian motion.
Household Wealth Dynamics

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Shares of Total Wealth

Let $\theta_i(t)$ be share of total wealth held by household $i$:

$$\theta_i(t) = \frac{x_i(t)}{x_1(t) + \cdots + x_N(t)}$$

Let $\theta_{\text{max}}(t)$ be share of total wealth held by wealthiest household:

$$\theta_{\text{max}}(t) = \max(\theta_1(t), \ldots, \theta_N(t))$$

Note that

- Wealth shares add up to one: $\theta_1(t) + \cdots + \theta_N(t) = 1$
- $0 < \theta_i(t) < 1$ for all $i = 1, \ldots, N$, and so $\theta_{\text{max}}(t) < 1$
The Model
Equilibrium Dynamics

Theorem (Wealth Dynamics Theorem)

If households face uninsurable idiosyncratic investment risk, then the share of the economy’s total wealth held by the wealthiest single household, $\theta_{\text{max}}$, satisfies

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \theta_{\text{max}}(t) \, dt = 1, \quad \text{a.s.}$$

- Economy’s wealth eventually concentrates at very top of distribution
  - Wealth distribution is not stationary
  - Limit of time-average of $\theta_{\text{max}}(t)$, not of $\theta_{\text{max}}(t)$ itself
- Similar result obtains in more general settings
  - Time-varying abilities and patience
  - As long as wealthy households do not have lower expected performance
Wealth Dynamics Theorem

Figure: The share of total wealth held by the wealthiest household.
Equality of Opportunity and Ability

How surprising is this result?

All households in the economy are essentially identical:

- Same abilities and opportunities
  - Labor income, expected returns of individual-specific assets
- Same patience
  - Preferences for consumption over time

The implication is that luck alone, in the form of high realized investment returns, generates this extreme divergence.
No Redistributive Mechanisms

- No redistributive mechanisms, broadly defined, causes concentration
  - Explicit: government tax or fiscal policies
  - Implicit: limited intergenerational transfers

- Crucial distinction between this and all other setups
  - Champenowne (1953), Gabaix (2009), Benhabib, Bisin, & Zhu (2011)

- What is a redistributive mechanism, exactly?
  - Wealth dynamics for each household $i$:
    \[ dx_i(t) = \Lambda x_i(t) dt + \Gamma x_i(t) dB_i(t) \]
  - Redistributive mechanism means lower $\Lambda$ for wealthier households
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Implications

- Redistributive mechanisms play crucial stabilizing role in economy
  - Many ways this can occur, but stability requires *something*

- What are the important real-world redistributive mechanisms?

- Why do wealthy households grow more slowly than poor households?
  - Taxes, fiscal policy, intergenerational transfers, psychology
  - A possible stabilizing role for redistributive income/estate taxes?
Parameterizing the Model

Wealth dynamics for each household $i$:

$$dx_i(t) = \Lambda x_i(t) \, dt + \Gamma x_i(t) \, dB_i(t)$$

Consequently, all that matters for simulations is

$$\Gamma = \frac{\alpha - r}{\gamma \sigma}$$

Benchmark parameterization:

- $N = 1,000,000$ households, $\alpha = 0.07$, $\sigma = 0.2$, and $r = 0.03$, $\gamma = 2$

- These values yield $\Gamma \approx 0.1$
Benchmark Parameterization: Wealth Shares

Figure: The shares of total wealth held by the wealthiest 1% (black), the wealthiest 1-5% (dotted red), and the wealthiest 5-10% (dashed green). ($\Gamma = 0.1$)
Benchmark Parameterization: Gini Coefficient

Figure: The Gini coefficient of the economy for $\Gamma = 0.1$. 
**Key Role of Idiosyncratic Investment Risk**

- Uninsurable investment risk and rate of wealth concentration linked
  - Increased exposure to idiosyncratic risk leads to faster wealth concentr.

- Recall, all that matters for simulations is
  \[ \Gamma = \frac{\alpha - r}{\gamma \sigma} \]

- Benchmark parameterization: \( \Gamma = 0.1 \)

- Two alternative parameterizations
  - High exposure to idiosyncratic investment risk: \( \Gamma = 0.2 \)
  - Low exposure to idiosyncratic investment risk: \( \Gamma = 0.05 \)
Benchmark Parameterization

Figure: The shares of total wealth held by the wealthiest 1% (black), the wealthiest 1-5% (dotted red), and the wealthiest 5-10% (dashed green). ($\Gamma = 0.1$)
More Exposure to Investment Risk

Figure: The shares of total wealth held by the wealthiest 1% (black), the wealthiest 1-5% (dotted red), and the wealthiest 5-10% (dashed green). (Γ = 0.2)
Less Exposure to Investment Risk

Figure: The shares of wealth held by the wealthiest 1% (black), the wealthiest 1-5% (dotted red), and the wealthiest 5-10% (dashed green). ($\Gamma = 0.05$)
Gini Coefficients and Exposure to Investment Risk

**Figure:** The Gini coefficient of the economy for $\Gamma = 0.1$ (solid black line), $\Gamma = 0.05$ (dotted red line), and $\Gamma = 0.2$ (dashed green line).
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Summary

- Three key elements to the model:
  1. Heterogeneous households that face idiosyncratic investment risk
  2. No redistributive mechanisms
  3. Forward-looking households that behave optimally

- In this setting, equilibrium distribution of wealth is not stationary
  - Increasing right-skewness, eventually full concentration at the top

- Instability and concentration a consequence of luck alone
  - Different realized investment returns
Implications and Extensions

- In the presence of uninsurable investment risk, the natural tendency of wealth is to concentrate
  - Luck, in the form of different realized investment returns
- Redistributive mechanisms play crucial stabilizing role in economy
  - Which mechanisms are most important in modern economies?
    - Intergenerational transfers, taxes, something else?
- Analytic techniques used in paper easily extended to stable economies
  - Household-by-household description of equilibrium wealth distribution
The End

Thank You